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Large Deviations, Guerra's and A.S.S. Schemes, and the Parisi Hypothesis

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We investigate the problem of computing

$$\lim_{N\to\infty} \frac{1}{aN} \log E Z_N^a$$

for any value of a, where Z_N is the partition function of the celebrated Sherrington-Kirkpatrick (SK) model, or of some of its natural generalizations. This is a natural "large deviation" problem. Its study helps to get a fresh look at some of the recent ideas introduced in the area, and raises a number of natural questions. We provide a complete solution for $a \geq 0$.

KEY WORDS: Mean field models, Spin glasses, Parisi solution

1. INTRODUCTION

The famous formula of Parisi to compute

$$\lim_{N\to\infty} \frac{1}{N} E \log Z_N \tag{1.1}$$

(where Z_N is the partition function of the SK model at a given temperature) has recently been rigorously justified. A next step would be the computation, for any value of a, of

$$\lim_{N\to\infty} \frac{1}{aN} \log E Z_N^a, \tag{1.2}$$

which, when $a = 0$, is naturally interpreted as (1.1) (see (1.3) below). As will be apparent in the next section, this is a kind of "large deviation" question. Another

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motivation for studying the quantity (1.2) is that it is related to the (mathematically unsound) "replica method" of the physicists. This method relies on the fact that at given N one has

$$\lim_{a\to 0} \frac{1}{aN} \log E Z_N^a = \frac{1}{N} E \log Z_N. \tag{1.3}$$

One then "computes" $\lim_{N\to\infty} \frac{1}{aN} \log E Z_N^a$ for $a \in \mathbb{N}^*$, one makes a (very clever) guess as how the computation should extend for $a > 0$ and one performs the interversion of the limits $a \to 0$ and $N \to \infty$ while keeping as many fingers crossed as feasible. This is explained in Ref. 8. On the other hand, as will be shown in Sec. 7, the value of the limit (1.2) when, say, $a = \frac{1}{2}$ (or more generally, $0 < a < 1$) is obtained by a construction of the same nature as the one needed to compute the limit (1.1) and it seems very difficult to justify this value as an extrapolation of the case $a \in \mathbb{N}^*$. In fact, the author now feels somewhat confident to risk the opinion that the replica method has succeeded in finding the value of the limit (1.1) not because it has any soundedness at all, but rather because of the extraordinary talent and inventiveness of G. Parisi. In other words, it does not seem in this case that there is more to the replica method than the Parisi Ansatz. The author certainly feels that attempting to provide a sound justification for the replica method is not currently a fruitful line of research.

The main new result of the paper is the computation of the limit (1.2) for all $a > 0$. The goal of the paper is more ambitious, as we attempt to review (in their natural adaptation to the present setting) a number of the ideas recently introduced in the area. This will be done at a leisurely pace, in the sense that we will explore several avenues that look very natural, even though they currently end in a cul de sac. This is because we believe that they are promising and deserve future investigation.

We start the paper by the computation of the limit (1.2) (and of the distribution of the weights of the configurations) in the case of Derrida's Random Energy Model. Of course the REM is a toy model, and all questions about it can be settled with elementary probability methods. Yet, this study is instructive. In particular, it provides motivation for introducing the Poisson-Dirichlet distributions $PD(m, a)$ (the distribution $PD(m, 0)$ is denoted by Λ_m in Ref. 12).

In the rest of the paper we study the "multi-p-spin model" (with Hamiltonian given by the formula (3.2) below). The Hamiltonian $H_N(\boldsymbol{\sigma})$ is a Gaussian r.v. and these variables are correlated in a way that

$$E H_N(\boldsymbol{\sigma}^1) H_N(\boldsymbol{\sigma}^2) = N\xi\left(\frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2\right) \tag{1.4}$$

for a certain function ξ. In the case of the SK model at inverse temperature β, $\xi(x) = \beta x^2/2$. In Sec. 3 we compute the limit (1.2) under a "high-temperature" condition (i.e. well inside the domain of validity of the replica-symmetric solution). Using an idea of R. Latala, we give a particularly efficient proof.

In Sec. 4 we develop in our setting the Aizenman-Sims-Starr (A.S.S.) scheme. This very beautiful scheme gives a representation of the limit (1.2) in a manner that is, in some sense, absolutely natural and canonical. The limit is represented as the infimum of a certain quantity over a family of random weights (and other parameters). These random weights have a precise physical meaning. There is obviously a large gap in our current understanding because at present we do not see how the problem of computing this infimum could be approached. On the other hand specific choices of weights (and of the other parameters) allows one to get upper bounds on the limit (1.2) when $a < 1$ (and lower bounds when $a > 1$).

In Sec. 5 we develop the notion of Poisson-Dirichlet cascades (based on distributions $PD(m, a)$ for a given value of a and various values of $m > a$) and we use the A.S.S. scheme to obtain for $a < 1$ upper bounds for the quantity (1.2) (when $a > 1$ we obtain instead a lower bound which is simply the replica-symmetric solution). We conjecture that these bounds give in fact the exact value of the limit for all values of a.

F. Guerra has invented a famous scheme to bound the quantity (1.1), a scheme that played a fundamental role on the later developments of the theory.[(4)] This scheme can be interpreted (using Poisson-Dirichlet cascades based on the distributions $PD(m, 0)$) as a special case of the A.S.S. scheme in its original formulation, but F. Guerra explained to the author that he invented his scheme based on purely analytical considerations. His scheme can be extended to the case (1.2), as we explain in Sec. 6. It is rather interesting that the upper bounds obtained when $a < 0$ seem more general than the upper bounds obtained combining Secs. 4 and 5. This is because Guerra's scheme allows values of m that are negative, while the distribution $PD(m, a)$ exists only for $a < m < 1$, $m > 0$. Of course one can expect that these seemingly more general upper bounds are in fact the same as those obtained previously, but this is by no means obvious. It is a purely analytical problem to decide this. Similarly, the lower bounds obtained when $a > 1$ with Guerra's scheme are seemingly more general than the bound obtained through the A.S.S. scheme (which in this case corresponds to the replica-symmetric solution), but in this case a rather deep recent analytical result of D. Panchenko[(10)] shows that they are in fact the same.

The Hamiltonian of the p-spin interaction model is a superposition of p-spin interactions. When only terms with p even are present in this superposition, we explain in Sec. 7 how to modify the proof of Ref. 13 to obtain the limit (1.2) when $0 < a < 1$. The modifications are minor, and the arguments are actually somewhat simpler than in the case $a = 0$.

In Sec. 8 we turn to the case $a > 1$. The approach we propose succeeds only for $1 < a \leq 2$. The reason we cannot make this approach work is that certain natural bounds involve optimization over certain parameters, and we do not know how to choose these parameters efficiently. These bounds are similar in nature (but somewhat simpler) to the bounds occurring in what seems the natural approach to the fundamental problems of ultrametricity and chaos.[15] It seems to the author that these problems are all connected, and constitute the central remaining mystery. This is what motivates the inclusion of this largely unsuccessful approach, despite the fact that another approach succeeds in Sec. 9 to compute the limit (1.2) for $a > 1$. Roughly speaking the argument of Sec. 9 uses (as inspired by the celebrated Ghirlanda-Guerra identities) the information that the function

$$\beta \mapsto (aN)^{-1} \log E \left(\sum_{\boldsymbol{\sigma}} \exp \beta H_N(\boldsymbol{\sigma}) \right)^a$$

is convex in β, an information that, when $a > 1$, turns out to be of immense power. Despite the fact that the proof is not really complicated, the author must confess that he does not really understand why it works; but of course, he must confess that he does not really understand either why the Ghirlanda-Guerra identities are true.

In Sec. 10 we investigate the limit (1.2) when $a < 0$ in the special case where there is no external field. We give arguments supporting the fact (conjectured by Dotsenko, Franz and Mézard) that this limit coincides with the limit when $a = 0$. Our arguments would actually provide a proof of this fact if one knew (as is widely believed to be the case) that the system "decomposes in pure states."

Finally, in Sec. 11, we briefly investigate the spherical model. This model is of interest because the Parisi functional is analytically simpler than it is in the Ising model considered in the previous sections. We are able to show, when $a < 0$, that the bounds through Guerra's scheme coincide with the bounds obtained through the A.S.S. scheme and Poisson-Dirichlet cascades.

2. THE REM

In this section, and until Sec. 10, we consider only Ising spins, and the configuration space is $\Sigma_N = \{-1, 1\}^N$. For $\boldsymbol{\sigma} \in \Sigma_N$, we consider i.i.d. standard Gaussian r.v. $H_N(\boldsymbol{\sigma})$ with $EH_N(\boldsymbol{\sigma})^2 = N/2$, the normalization of Ref. 12. Throughout the paper, we do not use the minus signs customary in physics, so that, at inverse temperature β, the partition function is given by

$$Z_N = \sum_{\boldsymbol{\sigma}} \exp \beta H_N(\boldsymbol{\sigma}). \tag{2.1}$$

We recall the classical fact (Ref. 12, Proposition 1.1.5) that

$$\lim_{N\to\infty} \frac{1}{N} E \log Z_N = \begin{cases} \log 2 + \dfrac{\beta^2}{4} & \text{if } \beta \le 2\sqrt{\log 2} \\ \beta\sqrt{\log 2} & \text{if } \beta \ge 2\sqrt{\log 2}. \end{cases}$$

Theorem 2.1.
a) If $a\beta \le 2\sqrt{\log 2}$*, we have*

$$\lim_{N\to\infty} \frac{1}{Na} \log E Z_N^a = \lim_{N\to\infty} \frac{1}{N} E \log Z_N. \tag{2.2}$$

b) If $a\beta \ge 2\sqrt{\log 2}$*, we have*

$$\lim_{N\to\infty} \frac{1}{Na} \log E Z_N^a = \max\left(\beta\sqrt{\log 2} + \frac{(a\beta - 2\sqrt{\log 2})^2}{4a}, \lim_{N\to\infty} \frac{1}{N} E \log Z_N\right). \tag{2.3}$$

The case (2.2) includes the case $a < 0$. Throughout the paper when $a = 0$ we interpret $(Na)^{-1} \log E Z_N^a$ as $N^{-1} E \log Z_N$.

Proof: The main comment about the upcoming proof is that it is very much harder than what one would expect (which is precisely why we write it despite the fact that it is somewhat tedious). This proof occupies several pages of hard work, and we urge the reader that is not specifically interested in this type of arguments to skip it. The arguments of the other sections of the paper are fortunately different from the bookkeeping required here.

First, we note that by Hölder's inequality

$$\text{The map } a \mapsto a^{-1} \log E Z_N^a \text{ is non-decreasing.} \tag{2.4}$$

We recall that for a normal r.v. g with $Eg^2 = \sigma^2$ we have, for $t \ge 0$

$$\frac{1}{L(1 + t/\sigma)} \exp\left(-\frac{t^2}{2\sigma^2}\right) \le P(g \ge t) \le \exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{2.5}$$

Here, as well as in the rest of the paper, L denotes a universal constant, not necessarily the same at each occurrence. Thus, if $c_N(t) = \sqrt{N}/(L(\sqrt{N} + t))$ we have, for each $\boldsymbol{\sigma}$,

$$P(H_N(\boldsymbol{\sigma}) \ge N\sqrt{\log 2} + t) \ge c_N(t) \exp\left(-\frac{(t + N\sqrt{\log 2})^2}{N}\right)$$
$$= 2^{-N} c_N(t) \exp\left(-2t\sqrt{\log 2} - t^2/N\right). \tag{2.6}$$

When we consider M independent events in a measure space, each of probability ε, when $\varepsilon M \le \frac{1}{2}$ the probability that at least one of them occurs is

$$1 - (1-\varepsilon)^M \ge 1 - \exp(-\varepsilon M) \ge \frac{\varepsilon M}{2}$$

and thus from (2.6) we see that for $t > 0$

$$P(\max_{\boldsymbol{\sigma}} H_N(\boldsymbol{\sigma}) \ge N\sqrt{\log 2} + t) \ge \frac{c_N(t)}{2} \exp\left(-2t\sqrt{\log 2} - \frac{t^2}{N}\right). \qquad (2.7)$$

For $a > 0$, we have, making the change of variable $u = \exp \beta a(N\sqrt{\log 2} + t)$, that

$$E Z_N^a = \int_0^\infty P\big(Z_N^a \ge u\big)\, du$$
$$\ge \beta a \exp(\beta a N \sqrt{\log 2}) \int_0^\infty P(\log Z_N \ge \beta(N\sqrt{\log 2} + t)) \exp(\beta a t)\, dt.$$

Now $\log Z_N \ge \beta \max_{\boldsymbol{\sigma}} H_M(\boldsymbol{\sigma})$ and using (2.7) we see that

$$E Z_N^a \ge \beta a \exp(\beta a N \sqrt{\log 2}) \int_0^\infty \frac{c_N(t)}{2} \exp\left((\beta a - 2\sqrt{\log 2})t - \frac{t^2}{N}\right) dt.$$

When $\beta a \ge 2\sqrt{\log 2}$, this easily yields

$$\liminf_{N\to\infty} \frac{1}{Na} \log E Z_N^a \ge \beta\sqrt{\log 2} + \frac{(\beta a - 2\sqrt{\log 2})^2}{4a}.$$

Combining with (2.4) this proves that the left hand side of (2.3) is bounded below by the right hand side.

We turn to the reverse inequality. As the previous argument shows, the trouble arises with the largest of the quantities $H_N(\boldsymbol{\sigma})$, and we have to control these. Since $P(H_N(\boldsymbol{\sigma}) \ge t) \le \exp(-t^2/N)$, the probability that at least k of the r.v. $H_N(\boldsymbol{\sigma})$ are $\ge v_k$ is at most

$$\binom{2^N}{k} \exp\left(-\frac{k v_k^2}{N}\right) \le \left(\frac{e 2^N}{k}\right)^k \exp\left(-\frac{k v_k^2}{N}\right). \qquad (2.8)$$

Thus, if $U(k)$ denotes the kth-largest term of the sequence $(\exp \beta H_N(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}}$ we have

$$P(U(k) \ge \exp \beta v_k) \le \left(\frac{e 2^N}{k}\right)^k \exp\left(-\frac{k v_k^2}{N}\right). \qquad (2.9)$$

Making the choice $v_k = \sqrt{N \log(e 2^N)} + t$, we get, since $\sqrt{N\log(e2^N)} \ge N\sqrt{\log 2}$,

$$P(U(k) \ge \exp \beta t \exp \beta \sqrt{N \log(e 2^N)}) \le \exp\left(-2kt\sqrt{\log 2} - \frac{kt^2}{N}\right). \qquad (2.10)$$

Now, for any $u_0 > 0$ we have

$$EU(k)^a = \int_0^\infty P(U(k)^a \geq u)\,du \leq u_0 + \int_{u_0}^\infty P(U(k)^a \geq u)\,du. \tag{2.11}$$

Taking $u_0 = \exp a\beta\sqrt{N\log(e2^N)}$, assuming $a \geq 0$, making the change of variables $u = u_0 \exp a\beta t$ and using (2.10), we get that

$$EU(k)^a \leq u_0 \left(1 + a\beta \int_0^\infty \exp\left(a\beta t - kt2\sqrt{\log 2} - \frac{kt^2}{N}\right) dt\right). \tag{2.12}$$

This yields

$$\limsup_{N\to\infty} \frac{1}{Na} \log EU(k)^a \leq \begin{cases} \beta\sqrt{\log 2} & \text{if } a\beta \leq 2k\sqrt{\log 2} \\ \beta\sqrt{\log 2} + \dfrac{(a\beta - 2k\sqrt{\log 2})^2}{4ak^2} & \text{if } a\beta \geq 2k\sqrt{\log 2}. \end{cases} \tag{2.13}$$

One should observe that the right-hand side is smaller that the right-hand side of (2.3).

Consider now an integer n, to be determined later, and

$$V(n) = V_N(n) = Z_N - (U(1) + \cdots + U(n)).$$

That is, we remove the n largest terms in the sum $Z_N = \sum \exp \beta H_N(\boldsymbol{\sigma})$. Since $V(n) = \sum_{n<k\leq 2^N} U(k)$, we see from (2.9) that for any numbers v_k, $1 \leq k \leq 2^N$, we have

$$P\left(V(n) \geq \sum_{k>n} \exp \beta v_k\right) \leq \sum_{k>n} P\left(U(k) \geq \exp \beta v_k\right)$$
$$\leq \sum_{k>n} \left(\frac{e2^N}{k}\right)^k \exp\left(-\frac{kv_k^2}{N}\right).$$

Making the choice

$$v_k = \sqrt{N \log \frac{e2^N}{k}} + t,$$

and setting

$$c(n) = c_N(n) = \sum_{k>n} \exp \beta \sqrt{N \log \frac{e2^N}{k}},$$

we get

$$P(V(n) \geq c(n) \exp \beta t) \leq \sum_{k>n} \exp\left(-2kt\sqrt{\frac{1}{N}\log\frac{e2^N}{k}} - \frac{kt^2}{N}\right).$$

Consider $u_0 = c(n)^a \exp a\beta$. Proceeding as in (2.11), (2.12) we get

$$EV(n)^a \le u_0 \left(1 + a\beta \int_1^\infty \sum_{k>n} \exp\left(a\beta t - 2kt\sqrt{\frac{1}{N}\log\frac{e2^N}{k}} - \frac{kt^2}{N}\right) dt\right). \tag{2.14}$$

When $k < 2^{N/2}$, we have

$$2kt\sqrt{\frac{1}{N}\log\frac{e2^N}{k}} \ge 2kt\sqrt{\frac{1}{N}\log 2^{N/2}} \ge kt,$$

while if $k > 2^{N/2}$ and $t \ge 1$, we have $\frac{kt^2}{N} \ge 2^{N/2}\frac{t}{N}$. Thus, we see that if we take n (independent of N) such that $n \ge a\beta + 1$, we have

$$EV(n)^a \le Ku_0,$$

where K is independent of N and thus

$$\limsup_{N\to\infty} \frac{1}{Na} \log EV(n)^a \le \limsup_{N\to\infty} \frac{1}{N} \log c_N(n). \tag{2.15}$$

Now we try to find an upper bound for $c(n)$. First let us assume that $\beta > 2\sqrt{\log 2}$. We write

$$\sqrt{\log\frac{e2^N}{k}} = \sqrt{N}\sqrt{\log 2 - \frac{1}{N}\log\frac{k}{e}} \le \sqrt{N\log 2}\left(1 - \frac{\log k/e}{2N\log 2}\right),$$

and thus

$$c(n) \le \exp N\beta\sqrt{\log 2}\sum_{k>n}\exp\left(-\beta\frac{\log k/e}{2\sqrt{\log 2}}\right) \le K\exp N\beta\sqrt{\log 2}.$$

Here and everywhere, K denotes a quantity that does not depend on N, and that need not be the same at each occurrence. Hence we have

$$\limsup_{N\to\infty}\frac{1}{N}\log c_N(n) \le \beta\sqrt{\log 2}. \tag{2.16}$$

Next, let us assume that $\beta \le 2\sqrt{\log 2}$. In that case, for $0 \le u \le 1$ and $\varepsilon > 0$, the contribution to $c(n)$ of the terms for which $2^{Nu} \le k \le 2^{N(u+\varepsilon)}$ is at most

$$2^{N(u+\varepsilon)}\exp\beta\sqrt{N\log e2^{N(1-u)}} \le \exp(N(u\log 2 + \beta\sqrt{1-u}\sqrt{\log 2} + \varepsilon\log 2) + K).$$

The maximum over u of $u\log 2 + \beta\sqrt{1-u}\sqrt{\log 2}$ is obtained for $u = 1 - \beta^2/(4\log 2)$ and is equal to $\log 2 + \beta^2/4$. It follows easily that

$$\limsup_{N\to\infty}\frac{1}{N}\log c_N(n) \le \log 2 + \frac{\beta^2}{4}, \tag{2.17}$$

and combining with (2.16) and (2.15) we see that

$$\limsup_{N\to\infty} \frac{1}{Na} \log E V(n)^a \le \lim_{N\to\infty} \frac{1}{N} E \log Z_N. \tag{2.18}$$

Now we have

$$Z_N^a \le K \left(\sum_{k\le n} U(k)^a + V(n)^a \right),$$

so that (2.13) and (2.18) prove that

$$\limsup_{N\to\infty} \frac{1}{Na} \log Z_N^a \le \lim_{N\to\infty} \frac{1}{N} E \log Z_N$$

when $a\beta \le 2\sqrt{\log 2}$, while

$$\limsup_{N\to\infty} \frac{1}{Na} \log Z_N^a \le \max\left(\beta\sqrt{\log 2} + \frac{(a\beta - 2\sqrt{\log 2})^2}{4a}, \lim_{N\to\infty} \frac{1}{N} E \log Z_N \right)$$

if $a\beta > 2\sqrt{\log 2}$. This finishes the proof of (2.3), and also shows that when $a > 0$ the left hand side of (2.2) is bounded by the right hand side. Using (2.4), all is left is to obtain a lower bound of the left-hand side of (2.2) when $a < 0$. Setting $b = -a$, we have to get an upper bound on

$$\begin{aligned} E Z_N^a = E Z_N^{-b} &= \int_0^\infty P\big(Z_N^{-b} \ge u\big)\, du \\ &= b\beta \int_{-\infty}^\infty P(Z_N \le \exp \beta t) \exp(-b\beta t)\, dt, \end{aligned} \tag{2.19}$$

using the change of variable $u = \exp(-b\beta t)$. If g denotes a standard Gaussian r.v., we have

$$P(Z_N \le \exp \beta t) \le P(\forall \boldsymbol{\sigma}, H_N(\boldsymbol{\sigma}) \le t) = P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}} \right)^{2^N}$$

and thus

$$E Z_N^a \le \mathrm{I} + \mathrm{II} + \mathrm{III} \tag{2.20}$$

where

$$\mathrm{I} = b\beta \int_{-\infty}^{0} P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}}\right)^{2^N} \exp(-b\beta t)\,dt$$

$$\mathrm{II} = b\beta \int_{0}^{N\sqrt{\log 2}} P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}}\right)^{2^N} \exp(-b\beta t)\,dt \tag{2.21}$$

$$\mathrm{III} = b\beta \int_{N\sqrt{\log 2}}^{\infty} \exp(-b\beta t)\,dt = \exp(-b\beta N\sqrt{\log 2}).$$

Using that for $t < 0$ we have

$$P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}}\right) \le \frac{1}{2}\exp\left(-\frac{t^2}{N}\right),$$

we see that for large N we have $\mathrm{I} \le 1$. Also,

$$\mathrm{II} \le b\beta N\sqrt{\log 2} \sup_{0\le t\le N\sqrt{\log 2}} P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}}\right)^{2^N} \exp(-b\beta t). \tag{2.22}$$

Now, using (2.5), we have

$$P\left(g \le \frac{\sqrt{2}t}{\sqrt{N}}\right) \le 1 - \frac{1}{L(1+t/\sqrt{N})}\exp\left(-\frac{t^2}{N}\right)$$

$$\le \exp\left(-\frac{1}{L(1+t/\sqrt{N})}\exp\left(-\frac{t^2}{N}\right)\right)$$

so that, for $t \le N\sqrt{\log 2}$ we have

$$P\left(g \le \frac{\sqrt{2}t}{N}\right)^{2^N} \le \exp\left(-\frac{2^N}{L(1+t/\sqrt{N})}\exp\left(-\frac{t^2}{N}\right)\right)$$

$$\le \exp\left(-\frac{2^N}{L\sqrt{N}}\exp\left(-\frac{t^2}{N}\right)\right)$$

$$\le \exp\left(-\frac{1}{L\sqrt{N}}\exp 2v\sqrt{\log 2}\right),$$

making the change of variable $t = N\sqrt{\log 2} - v$, and hence

$$\mathrm{II} \le b\beta N\sqrt{\log 2}\exp(-b\beta N\sqrt{\log 2}) \sup_{v>0} \exp\left(vb\beta - \frac{1}{L\sqrt{N}}\exp 2v\sqrt{\log 2}\right).$$

This shows that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathrm{II} \le -b\beta\sqrt{\log 2}$$

and thus

$$\lim_{N\to\infty} \frac{1}{Nb} \log E Z_N^a \le -\beta\sqrt{\log 2}.$$

This completes the argument when $\beta \ge 2\sqrt{\log 2}$. When $\beta \le 2\sqrt{\log 2}$, we need a better bound on

$$\mathrm{IV} = \int_0^{N\sqrt{\log 2}} P(Z_N \le \exp \beta t) \exp(-b\beta t)\, dt.$$

This is done by using the lower bound

$$Z_N \ge \exp \beta v \ \mathrm{card}\{\boldsymbol{\sigma}; H_N(\boldsymbol{\sigma}) \ge v\}$$

for any v, and estimating the probability that the last term is significantly smaller than its expectation using the tails of the binomial law. This is elementary and very, very tedious so it is left to the reader. □

It will gradually become apparent that when studying the limit (1.2), the right point of view is not to look at the typical structure of the Gibbs measure for the underlying probability but rather to make a change of density $Z_N^a / E Z_N^a$ in this underlying probability. We turn to the study of typical Gibbs weights of the configurations after this change of density

Theorem 2.2. *If $\beta < 2\sqrt{\log 2}$ and $a\beta < 2\sqrt{\log 2}$ all weights are infinitesimal.*

More formally, the expected value of the largest weight goes to zero as $N \to \infty$.

First Proof: We give this proof only when $a > 0$. The largest of the Gibbs weights is $U(1)/Z_N$, where $U(1) = \exp \beta \max H_N(\boldsymbol{\sigma})$. Denoting by E' expectation after change of density $Z_N^a / E Z_N^a$, we have

$$E'\left(\left(\frac{U(1)}{Z_N}\right)^a\right) = E\left(\frac{Z_N^a}{E Z_N^a}\left(\frac{U(1)}{Z_N}\right)^a\right) = \frac{E U(1)^a}{E Z_N^a}. \tag{2.23}$$

We recall that we have proved that

$$\lim_{N\to\infty} \frac{1}{Na} \log E U(1)^a \le \beta\sqrt{\log 2} < \lim_{N\to\infty} \frac{1}{Na} \log E Z_N^a. \tag{2.24}$$

The first inequality follows from (2.13) and the second from (2.3). Thus the quantity (2.23) is at most $\exp(-N/K)$, and $U(1)/Z_N$ is very small for E'. □

Second Proof: This proof works for $a \neq 1$, possibly negative. Define

$$p_N(\beta) = \frac{1}{Na} \log E Z_N^a,$$

so that

$$p_N'(\beta) = \frac{1}{N} \frac{1}{E Z_N^a} E \left(\sum_{\boldsymbol{\sigma}} H_N(\boldsymbol{\sigma}) \exp \beta H_N(\boldsymbol{\sigma}) Z_N^{a-1} \right).$$

We use that $H_N(\boldsymbol{\sigma})$ is a Gaussian r.v. with $E H_N(\boldsymbol{\sigma})^2 = N/2$, and Gaussian integration by parts to get

$$\begin{aligned} p_N'(\beta) &= \frac{1}{N E Z_N^a} \frac{\beta N}{2} \left(E Z_N^a + (a-1) E \left(Z_N^a \sum_{\boldsymbol{\sigma}} \frac{\exp 2\beta H_N(\boldsymbol{\sigma})}{Z_N^2} \right) \right) \\ &= \frac{\beta}{2} \left(1 + (a-1) E' \left(\sum_{\boldsymbol{\sigma}} G^2(\{\boldsymbol{\sigma}\}) \right) \right). \end{aligned} \tag{2.25}$$

Since

$$\lim_{N\to\infty} p_N(\beta) = \log 2 + \frac{\beta^2}{4},$$

we have

$$\lim_{N\to\infty} p_N'(\beta) = \frac{\beta}{2},$$

and from (2.25) this implies that $\lim_{N\to\infty} E'(\sum_{\boldsymbol{\sigma}} G^2(\{\boldsymbol{\sigma}\})) = 0$. □

Theorem 2.3. *If $\beta > 2\sqrt{\log 2}$ and $a\beta > 2\sqrt{\log 2}$, the largest Gibbs weight is nearly 1.*

The proof follows the idea of the first proof of Theorem 2.2 We use the estimates of Theorem 2.1 to see that for $k \geq 2$ the k-th largest weight is infinitesimal, and that the sum of all the n-largest weights for $\sqrt{2n} \geq a\beta + 1$ is infinitesimal. The details are left to the reader. □

The most interesting situation is the case not covered by Theorems 2.2 and 2.3, that is, $\beta > 2\sqrt{\log 2}$ and $a\beta < 2\sqrt{\log 2}$. Given $0 < m < 1$, we denote by $(u_j)_{j\geq 1}$ a non-increasing rearrangement of a realization of a Poisson point process of intensity measure $x^{-m-1}dx$ on $\mathbb{R}^+$.

Lemma 2.4. *If $a < m$ we have $E(\sum_{j\geq 1} u_j)^a < \infty$.*

Proof: If $a > 0$ we observe first that

$$E \sum_{j \geq 1} u_j 1_{\{u_j \leq 1\}} = \int_0^1 x^{-m} dx < \infty.$$

Moreover, since $a < 1$,

$$\left(\sum u_i 1_{\{u_i \geq 1\}}\right)^a \leq \sum u_i^a 1_{\{u_i \geq 1\}}$$

and, since $a < m$

$$E \sum u_i^a 1_{\{u_i \geq 1\}} = \int_1^\infty x^a x^{-m-1} dx < \infty.$$

If $a < 0$, we have much stronger facts, since, for $t > 0$,

$$P\left(\sum_{j \geq 1} u_j \leq t\right) \leq P(\forall j, u_j \leq t) = \exp\left(-\int_t^\infty x^{-m-1} dx\right)$$
$$= \exp\left(-\frac{t^{-m}}{m}\right).$$

□

In particular the sum $\sum_{j \geq 1} u_j$ is finite a.s. Let us consider the non-decreasing sequence $v_i = u_i / \sum_{j \geq 1} u_j$, and let us denote by $\mathcal{S}$ the (compact) set of sequences $(x_i)_{i \geq 1}$, $0 \leq x_i \leq 1$, $\sum_{i \geq 1} x_i \leq 1$, (x_i) non decreasing, so that $(v_i) \in \mathcal{S}$. The law of the random sequence (v_i) is a probability measure on the compact metric space $\mathcal{S}$, the so-called Poisson Dirichlet distribution. It is denoted Λ_m in Ref. 12 and $PD(m, 0)$ in Ref. 11.

Lemma 2.4 shows that we can make the change of density $(\sum u_j)^a / E(\sum u_j)^a$. Under this change of density, the sequence (v_j) has $PD(m, a)$ distribution (this defines $PD(m, a)$, a probability measure on $\mathcal{S}$).

The sequence of the Gibbs weights is the element (x_i) of $\mathcal{S}$ defined by $x_i = 0$ if $i \geq 2^N$ and x_i is the i-th largest of the numbers $G(\{\boldsymbol{\sigma}\})$ if $i \leq 2^N$.

Theorem 2.5. *If $a\beta < 2\sqrt{\log 2}$ and $\beta > 2\sqrt{\log 2}$, the law of the sequence of the Gibbs weights under the change of density $Z_N^a / E Z_N^a$ converges to $PD(m, a)$ for $m = 2\sqrt{\log 2}/\beta$.*

Proof: This proof is even more technical than the proof of Theorem 2.1, so it must be skipped by anyone not interested in proving results in this line. We follow the proof of the case $a = 0$ as given in Ref. 12, Theorem 1.2.1. Define a_N by $N a_N^2 = \log(2^N / \sqrt{N})$, and denote by $(h_i)_{i \leq 2^N}$ a non-increasing rearrangement of the numbers $H_N(\boldsymbol{\sigma}) - Na_N$. We define (keeping the dependence in N implicit)

$$e_i = \exp \beta h_i; \quad Z = Z_N = \sum_{i \le 2^N} e_i; \quad w_i = \frac{e_i}{Z}.$$

Consider a number b, and define

$$e_i^b = e_i 1_{\{h_i \ge b\}}; \quad Z_b = Z_{N,b} = \sum_{i \le 2^N} e_i^b; \quad w_i^b = \frac{e_i^b}{Z_b} \text{ (with } 0/0 = 0).$$

Define $w_i = w_i^b = 0$ if $i > 2^N$. Thus $(w_i) \in \mathcal{S}$, $(w_i^b) \in \mathcal{S}$.

Let us observe that $(\sum e_i)^a / E(\sum e_i)^a = Z^a / E Z^a$ and $(\sum e_i^b)^a / E(\sum e_i^b)^a = Z_b^a / E Z_b^a$. We want to prove that, given a continuous function f on $\mathcal{S}$

$$\forall \eta > 0, \quad \exists b, \quad \limsup_{N \to \infty} \left| E \frac{Z^a}{E Z^a} f((w_i)) - E \frac{Z_b^a}{E Z_b^a} f\big((w_i^b)\big) \right| \le 4\eta. \tag{2.26}$$

Once this is done, we conclude as in Ref. 12. We have, assuming without loss of generality that $|f| \le 1$,

$$\left| E \frac{Z^a}{E Z^a} f((w_i)) - E \frac{Z_b^a}{E Z_b^a} f\big((w_i^b)\big) \right| \le \mathrm{I} + \mathrm{II} \tag{2.27}$$

$$\mathrm{I} = E \frac{Z^a}{E Z^a} \left| f((w_i)) - f\big((w_i^b)\big) \right|$$

$$\mathrm{II} = E \left| \frac{Z^a}{E Z^a} - \frac{Z_b^a}{E Z_b^a} \right| \le \mathrm{III} + \mathrm{IV}$$

where

$$\mathrm{III} = E \frac{|Z^a - Z_b^a|}{E Z^a}$$

$$\mathrm{IV} = E Z_b^a \left| \frac{1}{E Z^a} - \frac{1}{E Z_b^a} \right| = E \left| \frac{E Z_b^a - E Z^a}{E Z^a} \right| \le E \left| \frac{Z^a - Z_b^a}{E Z^a} \right| = \mathrm{III}.$$

Since f is continuous, we can find $\varepsilon \ge 0$ such that

$$\sum_i \left| w_i - w_i^b \right| \le \varepsilon \Rightarrow \left| f((w_i)) - f\big((w_i^b)\big) \right| \le \eta.$$

Moreover the simple Lemma 1.2.4 of Ref. 12 shows that

$$\sum_i \left| w_i - w_i^b \right| \le \frac{Z - Z_b}{Z}.$$

Thus the left-hand side of (2.27) is bounded by

$$\eta + 2E \left(\frac{Z^a}{E Z^a} 1_{\{Z - Z_b \ge \varepsilon Z\}} \right) + 2E \left| \frac{Z^a - Z_b^a}{E Z^a} \right|,$$

and to prove (2.26) it suffices to show that we can find b such that for N large enough we have

$$E\left(\frac{Z^a}{EZ^a}1_{\{Z-Z_b\geq\varepsilon Z\}}\right)\leq\eta \tag{2.28}$$

$$E\frac{\left|Z^a-Z_b^a\right|}{EZ^a}\leq\eta. \tag{2.29}$$

Due to the lack of enthusiasm of the author about struggling with elementary (yet tough) computations, the proof will be given only when $a>0$. Let $X=(Z-Z_b)/Z$, so that $0\leq X\leq 1$. To prove (2.28), i.e. that $E'(1_{\{X\geq\varepsilon\}})\leq\eta$, it suffices to prove that $E'|X|^a\leq\eta\varepsilon^a$. If E' denotes expectation after change of density Z^a/EZ^a, we have

$$E'|X|^a=E\left(\frac{Z^a}{EZ^a}\left|\frac{Z-Z_b}{Z}\right|^a\right)=E\frac{|Z-Z_b|^a}{EZ^a}.$$

Also,

$$E\frac{\left|Z^a-Z_b^a\right|}{EZ^a}=E'(1-(1-X)^a)\leq aE'X.$$

It is the same to say that $E'X$ is small and $E'X^a$ is small (since $0\leq X\leq 1$) and thus (2.28) and (2.29) will follow from the fact that

$$\lim_{b\to-\infty}\limsup_{N\to\infty}\frac{E(Z-Z_b)^a}{EZ^a}=0, \tag{2.30}$$

or, equivalently

$$\lim_{b\to-\infty}\limsup_{N\to\infty}\frac{EZ_{N,b}^a}{EZ_N^a}=0, \tag{2.31}$$

where

$$Z_{N,b}\sum\{\exp\beta H_N(\boldsymbol{\sigma});H_N(\boldsymbol{\sigma})\geq Na_N+b\}.$$

The easy part is to prove that

$$EZ_N^a\geq\frac{1}{L}\exp a\beta Na_N. \tag{2.32}$$

This is because

$$\begin{aligned}P(H_N(\boldsymbol{\sigma})\geq Na_N)&\geq\frac{1}{L(1+Na_N/\sqrt{N})}\exp\left(-Na_N^2\right)\\&=\frac{2^{-N}\sqrt{N}}{L(1+\sqrt{N}a_N)}\geq\frac{2^{-N}}{L}\end{aligned}$$

since $a_N \le \sqrt{\log 2}$, and hence $\max H_N(\boldsymbol{\sigma}) \ge Na_N$ with probability $\ge 1/L$, and $Z_N^a \ge \exp a\beta N a_N$ with the same probability.

We now try to find upper bounds for $Z_{N,b}$. For a standard normal r.v. g we have that, for $t \ge 1$,

$$P(g \ge t) \le \frac{L}{t} \exp\left(-\frac{t^2}{2}\right)$$

so that for each $\boldsymbol{\sigma}$ we have, for $v_k \ge \sqrt{N}$

$$P(\operatorname{card}\{\boldsymbol{\sigma}; H_N(\boldsymbol{\sigma}) \ge v_k\} \ge k) \le \left(\frac{e2^N}{k}\right)^k \left(\frac{L\sqrt{N}}{v_k} \exp\left(-\frac{v_k^2}{N}\right)\right)^k. \tag{2.33}$$

This equation is similar to (2.8), but it is more precise. Consider the event Ω given by

$$\forall k \ge 1, \quad \operatorname{card}\{\boldsymbol{\sigma}; H_N(\boldsymbol{\sigma}) \ge v_k\} < k \tag{2.34}$$

so that by (2.33) we have

$$P(\Omega^c) \le \sum_{k \ge 1} \left(\frac{L2^N\sqrt{N}}{kv_k} \exp\left(-\frac{v_k^2}{N}\right)\right)^k. \tag{2.35}$$

On the event Ω, for any integer k_0, we have

$$Z_{N,b} \le k_0 \exp \beta(Na_N + b) + \sum_{k > k_0} \exp \beta v_k. \tag{2.36}$$

This is because $Z_{N,b}$ is a sum of terms that are all $\le \exp \beta(Na_N + b)$, and the k-th largest of these terms is $\le \exp \beta v_k$.

Given $t > 0$, let us take

$$v_k = Na_N - \frac{\log k}{2\sqrt{\log 2}} + \frac{t}{2\sqrt{\log 2}}. \tag{2.37}$$

We note that

$$v_k \ge v_{2^N} = Na_N - \frac{N}{2}\sqrt{\log 2} + \frac{t}{2\sqrt{\log 2}},$$

and since $a_N \simeq \sqrt{\log 2}$ for N large, for all k we have $v_k \ge N/L$. Also, we have

$$v_k^2 \ge N^2 a_N^2 - Na_N \left(\frac{t}{\sqrt{\log 2}} - \frac{\log k}{\sqrt{\log 2}}\right),$$

and since $a_N \le \sqrt{\log 2}$ we have

$$-v_k^2 \le -N^2 a_N^2 + N \log k - Nta_N/\sqrt{\log 2}.$$

Denoting by Ω_t the event (2.34) corresponding to the choice (2.37) of v_k, we see from (2.35), and since $v_k \geq N/L$ that

$$P(\Omega_t^c) \leq \sum_{k\geq 1}\left(\frac{L2^N\sqrt{N}}{kv_k}\exp\left(-Na_N^2+\log k-\frac{ta_N}{\sqrt{\log 2}}\right)\right)^k$$
$$\leq \sum_{k\geq 1}\left(\frac{LN}{v_k}\exp\left(-\frac{ta_N}{\sqrt{\log 2}}\right)\right)^k \leq L\exp\left(-\frac{ta_N}{\sqrt{\log 2}}\right) \quad (2.38)$$

for t large enough, but independent of N. Since $\beta > 2\sqrt{\log 2}$ and since for $c > 1$ we have $\sum_{k>k_0} k^{-c} \leq K(c)k_0^{-c}$ where $K(c)$ depends on c only, we see from (2.37) and (2.36) that on Ω_t we have

$$Z_{N,b} \leq Lk_0 \exp\beta(Na_N+b) + K(c)\exp\beta v_k$$
$$\leq K(c)k_0\exp\beta(Na_N+b)$$

whenever $k_0 = k_0(t)$ is such that $v_{k_0} \leq Na_N + b$. From (2.37) we see that we can take $k_0(t)$ about $\exp(t-2b\sqrt{\log 2})$ so that, with probability $\geq 1 - L\exp(-ta_N/\sqrt{\log 2})$ we have

$$Z_{N,b}^a \leq K(\beta)^a\exp(at-2ab\sqrt{\log 2})\exp a\beta(Na_N+b)$$
$$= K(\beta)^a\exp at\exp a\beta Na_N\exp ab(\beta-2\sqrt{\log 2}).$$

Since $a < 1$ we have $a_N/\sqrt{\log 2} > a$ for large N and thus

$$EZ_{N,b}^a \leq K\exp(a\beta Na_N)\exp ab(\beta-2\sqrt{\log 2})$$

where K is independent of N and b. Since $\beta > 2\sqrt{\log 2}$, this completes the proof of (2.31). □

Proposition 2.6. *Consider a sequence $(v_i)_{i\geq 1}$ with $PD(m,a)$ distribution. Then*

$$E\sum_{i\geq 1} v_i^2 = \frac{1-m}{1-a}.$$

This, and much more, is known.[11] Our proof, however, is of interest.

Proof: Consider $\beta > 2\sqrt{\log 2}$. We use (2.25) and the fact that, since $p_N(\beta) \to \beta\sqrt{\log 2}$, we have $p'_N(\beta) \to \sqrt{\log 2}$. We then use Theorem 2.5 to obtain the result. □

The author discovered new relations related to the distribution $PD(m, 0)$ (see Ref. 12, Prop. 1.2.8). Upon seeing then J. Pitman immediately extended them to $PD(m, a)$. Theorem 2.5 shows that in fact the proof of Ref. 12, Prop. 1.2.8 extends verbatim to the case $a \neq 0$.

3. THE REPLICA-SYMMETRIC SOLUTION

For $p \geq 1$, $i_1, \ldots, i_p \in \mathbb{N}$, consider independent Gaussian standard r.v. $g_{i_1 \ldots i_p}$. Consider numbers β_p, $p \geq 1$, and assume that

$$C = \sum_{p \geq 2} \beta_p^2 p^2 < \infty. \tag{3.1}$$

In this section we consider the Hamiltonian

$$H_N(\boldsymbol{\sigma}) = \sum_{p \geq 1} \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1, \ldots, i_p} g_{i_1 \ldots i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \tag{3.2}$$

where the summation is taken over all values $1 \leq i_1, \ldots, i_p \leq N$. Thus, for two configurations $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$, we have

$$\begin{aligned} \frac{1}{N} E H_N(\boldsymbol{\sigma}^1) H_N(\boldsymbol{\sigma}^2) &= \sum_{p \geq 1} \frac{\beta_p^2}{N^p} \sum_{i_1, \ldots, i_p} \sigma_{i_1}^1 \sigma_{i_1}^2 \cdots \sigma_{i_p}^1 \sigma_{i_p}^2 \\ &= \xi(R_{1,2}) \end{aligned} \tag{3.3}$$

where

$$R_{1,2} = \frac{1}{N} \sum_{i \leq N} \sigma_i^1 \sigma_i^2$$

and

$$\xi(x) = \sum_{p \geq 1} \beta_p^2 x^p. \tag{3.4}$$

Theorem 3.1. *There exists a number L with the following property. Assume that*

$$LC(1 + a^2) \leq 1. \tag{3.5}$$

Then, the equation

$$q = \frac{E(\mathrm{th}^2 Y \mathrm{ch}^a Y)}{E \mathrm{ch}^a Y} \tag{3.6}$$

where $Y = z\sqrt{\xi'(q)} + h$ and z is standard normal has a unique solution. If

$$Z_N = \sum_{\sigma} \exp\left(H_N(\boldsymbol{\sigma}) + h\sum_{i\le N}\sigma_i\right),$$

and if we denote by $\langle\cdot\rangle$ an average for the Gibbs measure with Hamiltonian $H_N(\boldsymbol{\sigma}) + h\sum_{i\le N}\sigma_i$, we have

$$E\left(\frac{Z_N^a}{EZ_N^a}\left\langle \exp\frac{N}{32}(R_{1,2}-q)^2\right\rangle\right) \le L, \tag{3.7}$$

and

$$\left|\frac{1}{aN}\log EZ_N^a - p\right| \le \frac{K}{N}, \tag{3.8}$$

where K does not depend on N and

$$p = \log 2 + \frac{1}{2}(\xi(1) - \xi'(q)) + \frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{\xi'(q)} + h)$$
$$-\frac{1}{2}(a-1)(q\xi'(q) - \xi(q)).$$

Proof: Condition (3.5) is a "high temperature hypothesis." The uniqueness of the solution of (3.6) is obtained by proving that under (3.5) the function

$$q \mapsto \frac{E(\mathrm{th}^2 Y \mathrm{ch}^a Y)}{E\mathrm{ch}^a Y}$$

is a contraction. This is tedious but straightforward (using of course integration by parts).

The rest of the proof we present is inspired by an unpublished paper of R. Latala (following ideas of Guerra and Toninelli[(6)]). Consider the interpolating Hamiltonian

$$H_t = \sqrt{t}H_N(\boldsymbol{\sigma}) + \sqrt{1-t}\sum_{i\le N} z_i\sigma_i\sqrt{\xi'(q)} + h\sum_{i\le N}\sigma_i,$$

where z_i are i.i.d. standard Gaussian, independent of r.v. in H_N. Let $Z_t = \sum_{\boldsymbol{\sigma}} \exp H_t(\boldsymbol{\sigma})$, and for a function $f(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$ of two configurations consider

$$\varphi(t) = E\left(\frac{Z_t^a}{EZ_t^a}\langle f(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\rangle_t\right),$$

where of course $\langle\cdot\rangle_t$ denotes an average with respect to Gibbs measure with Hamiltonian H_t (configurations with different superscripts being averaged independently). Let us write

$$A(l, l') = \xi(R_{l,l'}) - R_{l,l'}\xi'(q) + q\xi'(q) - \xi(q),$$

where of course $R_{l,l'} = N^{-1}\sum_{i\le N}\sigma_i^l\sigma_i^{l'}$, and for simplicity let us write E'_t for expectation after change of density Z_t^a/EZ_t^a, so that $\varphi(t) = E'_t(\langle f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2\rangle_t)$. The core of the proof is the formula

$$\begin{aligned}\frac{2}{N}\varphi'(t) = {} & 2E'_t(\langle f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)A(1,2)\rangle_t)\\ & + 4(a-2)E'_t(\langle f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)A(1,3)\rangle_t)\\ & + (a-2)(a-3)E'_t\langle f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)A(3,4)\rangle_t\\ & + a(1-a)E'_t(\langle f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\rangle_t)E'_t(\langle A(1,2)\rangle_t). \end{aligned} \tag{3.9}$$

This is obtained by differentiation and integration by part, a straightforward (but tedious) computation.

The function

$$u(x) = \xi(x) - x\xi'(q) + q\xi'(q) - \xi(q)$$

satisfies $u(q) = u'(q) = 0$ so

$$u(x) \le (x-q)^2 \sup\{|u''(y)| : |y| \le 1\} \le (x-q)^2 C \tag{3.10}$$

and thus

$$A(l, l') \le C(R_{l,l'} - q)^2.$$

By Hölder's inequality we have relations such as

$$\langle (R_{1,2}-q)^{2k}(R_{1,3}-q)^2\rangle \le \langle (R_{1,2}-q)^{2k+2}\rangle$$

and thus (3.9) implies that when $f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) = (R_{1,2}-q)^k$, we have

$$\varphi'(t) \le CN(8+5a+a^2)E'_t(\langle (R_{1,2}-q)^{k+2}\rangle_t),$$

and thus, by power expansion, for any $A > 0$ we have

$$\frac{d}{dt}E'_t(\langle \exp A(R_{1,2}-q)^2\rangle_t) \le CN(8+5a+a^2)E'_t(\langle (R_{1,2}-q)^2 \exp A(R_{1,2}-q)^2\rangle_t,$$

and hence

$$\frac{d}{dt}E'_t(\langle \exp(A - CtN(8+5a+a^2))(R_{1,2}-q)^2\rangle_t) \le 0,$$

and in particular

$$E\left(\frac{Z_N^a}{EZ_N^a}\langle\exp(A - CN(8+5a+a^2)(R_{1,2}-q)^2\rangle\right) \le E_0'(\langle\exp A(R_{1,2}-q)^2\rangle_0). \tag{3.11}$$

To finish the proof of (3.7) under (3.5) we show that the right-hand side of (3.11) is $\le L$ when $A = N/16$. This is because in that case the r.v. $\sigma_i^1\sigma_i^2 - q$ are independent, of expectation 0 by (3.6), and $|\sigma_i^1\sigma_i^2 - q| \le 2$, and the result is standard probability.

To prove (3.8) we consider now

$$\varphi(t) = \frac{1}{Na}\log EZ_t^a,$$

so that by differentiation and integration by parts we have

$$2\varphi'(t) = \xi(1) - \xi'(q) + (a-1)E_t'(\langle\xi(R_{1,2}) - R_{1,2}\xi'(q)\rangle_t)$$

and thus by (3.7) and (3.10) we have

$$|2\varphi'(t) - (\xi(1) - \xi'(q) + (a-1)(\xi(q) - q\xi'(q)))| \le \frac{K}{N}.$$

Hence

$$\left|\varphi(1) - \varphi(0) - \frac{1}{2}(\xi(1) - \xi'(q) + (a-1)(\xi(q) - q\xi'(q)))\right| \le K/N,$$

and this proves (3.8) since obviously

$$\varphi(0) = \log 2 + \frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{\xi'(q)} + h).$$ □

4. THE AIZENMAN-SIMS-STARR SCHEME

Since the paper[(1)] is very concise, it took me some time to explain its results to myself, and I will use this opportunity to try to explain to others what I understand.

If the limit

$$U = \lim_{N\to\infty}\frac{1}{Na}\log EZ_N^a$$

exists, it is reasonable to hope that for large N we will have

$$U \simeq \frac{1}{a}\left(\log EZ_N^a - \log EZ_{N-1}^a\right) \tag{4.1}$$

and we will evaluate this quantity using the cavity method. The Hamiltonians we consider are essentially those of (3.2), but when using the cavity method it is cleaner to remove the "diagonal terms" and to use instead

$$H_N(\boldsymbol{\sigma}) = \sum_{1\le p\le N} \beta_p \left(\frac{p!}{N^{p-1}}\right)^{1/2} \sum_{1\le i_1<\ldots<i_p\le N} g_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}. \tag{4.2}$$

Let us write $\boldsymbol{\varrho} = (\sigma_1, \ldots, \sigma_{N-1})$, and

$$H^1(\boldsymbol{\varrho}) = \sum_{1\le p\le N} \beta_p \left(\frac{p!}{N^{p-1}}\right)^{1/2} \sum_{1\le i_1<\ldots<i_p\le N-1} g_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{4.3}$$

$$A(\boldsymbol{\varrho}) = \sum_{1\le p\le N} \beta_p \left(\frac{p!}{N^{p-1}}\right)^{1/2} \sum_{1\le i_1<\ldots<i_{p-1}\le N-1} g_{i_1\ldots i_{p-1}N}\sigma_{i_1}\cdots\sigma_{i_{p-1}}, \tag{4.4}$$

so that

$$H_N(\boldsymbol{\sigma}) = H^1(\boldsymbol{\varrho}) + \sigma_N A(\boldsymbol{\varrho}). \tag{4.5}$$

Let us define $Z = \sum_{\boldsymbol{\varrho}} \exp(H^1(\boldsymbol{\varrho}) + h\sum_{i\le N-1}\sigma_i)$ and $w(\boldsymbol{\varrho}) = Z^{-1}\exp(H^1(\boldsymbol{\varrho}) + h\sum_{i\le N-1}\sigma_i)$. Then we have the obvious identity

$$Z_N^a = Z^a \left(\sum_{\boldsymbol{\varrho}} w(\boldsymbol{\varrho}) 2\mathrm{ch}(A(\boldsymbol{\varrho}) + h)\right)^a$$

so that

$$\log EZ_N^a = a\log 2 + \log EZ^a + \log E\frac{Z^a}{EZ^a}\left(\sum_{\boldsymbol{\varrho}} w(\boldsymbol{\varrho})\mathrm{ch}(A(\boldsymbol{\varrho}) + h)\right)^a. \tag{4.6}$$

Consider new standard independent Gaussian r.v. $g'_{i_1\ldots i_p}$ and

$$H^2(\boldsymbol{\varrho}) = \sum_{1\le p\le N-1} \beta_p \left(\frac{p!}{(N-1)^{p-1}} - \frac{p!}{N^{p-1}}\right)^{1/2} \sum_{1\le i_1<\cdots<i_p\le N-1} g'_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{4.7}$$

so that the joint distribution of $H^1 + H^2$ is the same as the distribution of H_{N-1} and thus

$$\begin{aligned} EZ_{N-1}^a &= E\left(\sum_{\boldsymbol{\varrho}} \exp\left(H^1(\boldsymbol{\varrho}) + H^2(\boldsymbol{\varrho}) + h\sum_{i\le N-1}\sigma_i\right)\right)^a \\ &= EZ^a E\frac{Z^a}{EZ^a}\left(\sum_{\boldsymbol{\varrho}} w(\boldsymbol{\varrho})\exp H^2(\boldsymbol{\varrho})\right)^a \end{aligned}$$

and

$$\log E Z_{N-1}^a = \log E Z^a + \log E \frac{Z^a}{E Z^a} \left(\sum_{\varrho} w(\varrho) \exp H^2(\varrho) \right)^a . \tag{4.8}$$

Comparing with (4.1) and (4.6),

$$aU \simeq a \log 2 + \log E' \left(\sum_{\varrho} w(\varrho) \mathrm{ch}(A(\varrho) + h) \right)^a$$
$$- \log E' \left(\sum_{\varrho} w(\varrho) \exp H^2(\varrho) \right)^a , \tag{4.9}$$

where E' denotes expectation after change of density $Z^a / E Z^a$. Since Z and the terms $w(\varrho)$ are defined in terms of the r.v. $g_{i_1 \dots i_p}, i_1 < \cdots < i_p < N$, the processes $A(\varrho)$ and $H^2(\varrho)$ are probabilistically independent of these weights, before and after the change of density. Also, from (4.4) one sees that if $R^-_{1,2} = N^{-1} \sum_{i<N} \sigma_i^1 \sigma_i^2$, one has

$$E A(\varrho^1) A(\varrho^2) \simeq \xi'(R^-_{1,2}) \tag{4.10}$$

and, defining

$$\theta(x) = x\xi'(x) - \xi(x),$$

that

$$E H^2(\varrho^1) H^2(\varrho^2) \simeq \theta(R^-_{1,2}), \tag{4.11}$$

where $\simeq$ means equality within terms of order $1/N$. Thus, we see from (4.9) that we should look for U of the form

$$U \simeq \log 2 + \frac{1}{a} \log E \left(\sum w_\alpha \mathrm{ch}(h + z_\alpha) \right)^a - \frac{1}{a} \log E \left(\sum w_\alpha \exp y_\alpha \right)^a , \tag{4.12}$$

where the processes $(z_\alpha)_{\alpha \in A}$, $(y_\alpha)_{\alpha \in A}$ are jointly Gaussian and

$$E z_\alpha z_\gamma = \xi'(q_{\alpha\gamma}); \quad E y_\alpha y_\gamma = \theta(q_{\alpha\gamma}) \tag{4.13}$$

for a certain function $q_{\alpha\gamma}$ on $A \times A$ with $q_{\alpha\alpha} = 1 - 1/N$, where $A = \{-1, 1\}^{N-1}$. The fact that we did not have exact equality in (4.10) and (4.11) is not a problem. One can find a small modification of the processes $A(\varrho)$ and $H^1(\varrho)$ (by adding suitable diagonal terms) such that one has exact equality, and make a further adjustment to ensure that $q_{\alpha\alpha} = 1$ rather than $q_{\alpha\alpha} = R^-(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = 1 - 1/N$. Rather than (4.1), we could also have argued that for given M, and N large we should have

$$U \simeq \frac{1}{aM} \big(\log E Z_N^a - \log E Z_{N-M}^a \big).$$

Rather that (4.12) we would then get that

$$U \simeq \log 2 + \frac{1}{aM} \log E\Big(\sum w_\alpha \prod_{j\le M} \mathrm{ch}(h + z_{j,\alpha})\Big)^a$$

$$-\frac{1}{aM} \log E\left(\sum w_\alpha \exp \sqrt{M} y_\alpha\right)^a, \tag{4.14}$$

where (y_α) are as in (4.13) and $(z_{j,\alpha})_{j\le M}$ are independent copies of the process (z_α) of (4.13). The arguments do not show that the distributions (w_α) should be the same in (4.12) and (4.14), but one sees no reason why they should be different; so, indeed this must be a remarkable distribution.

Let us now set

$$U_N = \frac{1}{a} \log E Z_N^a$$

and define

$$W^-(M) = \inf \Big\{ \log 2 + \frac{1}{aM} \log E\Big(\sum w_\alpha \prod_{j\le M} \mathrm{ch}(h + z_{j,\alpha})\Big)^a$$

$$- \frac{1}{aM} \log E\left(\sum w_\alpha \exp \sqrt{M} y_\alpha\right)^a \Big\},$$

where the infimum is over all possible choices of the set A, the function $q_{\alpha\gamma}$ (with $q_{\alpha\alpha} = 1$), the random weights (w_α), and where of course the $(z_{j,\alpha})$ and y_α are as in (4.14). We define $W^+(M)$ similarly, when the infimum is replaced by a supremum.

Proposition 4.1. *We have*

$$\liminf \frac{U_N}{N} \ge \sup_M W^-(M) \tag{4.15}$$

$$\limsup \frac{U_N}{N} \le \inf_M W^+(M). \tag{4.16}$$

Proof: Given M, for N large the previous analysis shows that $U_N - U_{N-M} \ge W^-(M) - \varepsilon_M$, where $\varepsilon_M \to 0$, and this obviously implies (4.15). The proof of (4.16) is similar. □

For our next argument it is better to go back to the Hamiltonian (3.2) "with the diagonal terms." We denote this Hamiltonian by H'_N to distinguish it from the Hamiltonian (4.3), and we write

$$U'_N = a^{-1} \log E {Z'_N}^a, \quad \text{where } \; Z'_N = \sum \exp H'_N(\boldsymbol{\sigma}).$$

First, we observe that

$$\lim_{N\to\infty}\left|\frac{U_N}{N}-\frac{U'_N}{N}\right|=0. \tag{4.17}$$

This is because (in distribution) we have $H'_N H_N + H''_N$ (where H''_N consists of the diagonal terms) and $EH''_N(\boldsymbol{\sigma})^2 \le K$, where K depends on p but not on N. We then use the bounds

$$Z_N \exp(-A) \le Z'_N \le Z_N \exp A$$

where $A = \max_{\boldsymbol{\sigma}} H''_N(\boldsymbol{\sigma})$. Given $\varepsilon > 0$, we have $P(|A| \ge \varepsilon N) \le 2^N \exp(-\varepsilon^2 N^2/K')$, where K' depends on p only, so that (using Hölder's inequality) one can show that the event $\{|A| \ge \varepsilon N\}$ is so small as being irrelevant. The details are left to the reader.

Proposition 4.2. *Assume that ξ is convex (e.g. $\beta_p = 0$ for p odd). Then, if $a \le 1$, we have*

$$\frac{U'_N}{N} \le W^-(N) \tag{4.18}$$

and if $a \ge 1$ we have

$$\frac{U'_N}{N} \ge W^+(N). \tag{4.19}$$

Corollary 4.3. *If $a \le 1$ and the function ξ is convex, we have*

$$\lim_{N\to\infty}\frac{U_N}{N} = \sup_M W^-(M) = \lim_{M\to\infty} W^-(M). \tag{4.20}$$

If $a \ge 1$, we have

$$\lim_{N\to\infty}\frac{U_N}{N} = \inf_M W^+(M) = \lim_{M\to\infty} W^+(M). \tag{4.21}$$

Proof: By (4.15), (4.17) and (4.18) we have

$$\begin{aligned}\sup_M W^-(M) &\le \liminf_N \frac{U_N}{N} \le \limsup_N \frac{U_N}{N}\\ &= \limsup_N \frac{U'_N}{N} \le \limsup_N W^-(N) \le \sup_M W^-(M),\end{aligned}$$

so that

$$\lim_N \frac{U_N}{N} = \lim_N \frac{U'_N}{N} = \sup_N W^-(N),$$

and, using (4.17) again,

$$\sup_N W^-(N) = \lim_N \frac{U'_N}{N} \le \liminf_N W^-(N) \le \limsup_N W^-(N) \le \sup_N W^-(N)$$

and thus $\lim W^-(N) = \sup_N W^-(N)$. This prove (4.20). The proof of (4.21) is similar. □

Proof of Proposition 4.2. Consider a set A, a function $(q_{\alpha\gamma})$ on A^2, with $q_{\alpha\alpha} = 1$, Gaussian processes (z_α), (y_α) as in (4.13) and independent copies $(z_{i,\alpha})$ of (z_α). Consider the Hamiltonian

$$H_t(\boldsymbol{\sigma}, \alpha) = \sqrt{t}(H_N(\boldsymbol{\sigma}) + \sqrt{N} y_\alpha) + \sqrt{1-t} \sum_{i \le N} z_{i,\alpha}\sigma_i + h \sum_{i \le N} \sigma_i$$

and

$$Z_t = \sum_{\alpha, \boldsymbol{\sigma}} w_\alpha \exp H_t(\boldsymbol{\sigma}, \alpha).$$

Let

$$\varphi(t) = \frac{1}{aN} \log E Z_t^a,$$

so that

$$2\varphi'(t) = \frac{1}{2N} \frac{1}{E Z_t^a} E \left(Z_t^{a-1} \sum_{\alpha, \boldsymbol{\sigma}} \left(\frac{1}{\sqrt{t}} (H_N(\boldsymbol{\sigma}) + \sqrt{N} y_\alpha) - \frac{1}{\sqrt{1-t}} \sum_{i \le N} z_{i,\alpha}\sigma_i \right) \exp H_t(\boldsymbol{\sigma}, \alpha) \right).$$

We integrate by parts, using that $E H_N(\boldsymbol{\sigma}^1) H_N(\boldsymbol{\sigma}^2) = N\xi(R_{1,2})$, $E y_\alpha y_\beta = \theta(q_{\alpha\beta})$, $E z_{i,\alpha} z_{i,\beta} = \xi'(q_{\alpha\beta})$, $E z_{i,\alpha} z_{j,\beta} = 0$ if $i \ne j$ and that $q_{\alpha\alpha} = 1$ to see that

$$\varphi'(t) = \frac{a-1}{2} E \left(\frac{Z_t^a}{E Z_t^a} \langle \xi(R_{1,2}) - R_{1,2}\xi'(q_{\alpha\beta}) + \theta(q_{\alpha\beta}) \rangle_t \right),$$

where

$$\langle f \rangle_t = Z_t^{-2} \sum w_\alpha w_\beta f(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \alpha, \beta) \exp(H_t(\boldsymbol{\sigma}^1, \alpha) + H_t(\boldsymbol{\sigma}^2, \beta)).$$

By convexity of ξ we have $\xi(R_{1,2}) - R_{1,2}\xi'(q_{\alpha\beta}) + \theta(q_{\alpha\beta}) \ge 0$ so that (when $a \le 1$) we have $\varphi(1) \le \varphi(0)$. Now

$$\varphi(1) = \frac{1}{aN} \log E Z_N^a + \frac{1}{aN} \log E \left(\sum_\alpha w_\alpha \exp \sqrt{N} y_\alpha \right)^a$$

$$\varphi(0) = \log 2 + \frac{1}{aN} \log E \left(\sum_{\alpha} w_\alpha \prod_{j \le N} \mathrm{ch}(h + z_{i,\alpha}) \right). \qquad \square$$

The problem with Corollary 4.3 is that one does not see easily how to compute $W^+(M)$ and $W^-(M)$. We conclude this section by a few remarks.

Lemma 4.4. *We have $W^+(M) \le W^+(1)$.*

Proof: Consider a set A, a function $q_{\alpha\beta}$ on A^2, with $q_{\alpha\alpha} = 1$, and (z_α), (y_α) as in (4.13). Consider independent copies $(y_{j,\alpha})$, $(z_{j,\alpha})$ of the processes (y_α) and (z_α). For $1 \le k \le M$ consider

$$w_{\alpha,k} = w_\alpha \prod_{j \le k-1} \mathrm{ch}(h + z_{j,\alpha}) \exp \sum_{k+1 \le j \le M} y_{j,\alpha}$$

so that

$$\frac{1}{aM} \log E \left(\sum w_\alpha \prod_{j \le M} \mathrm{ch}(h + z_{j,\alpha}) \right)^a - \frac{1}{aM} \log E \left(\sum w_\alpha \exp \sum_{j \le M} y_{j,\alpha} \right)^a$$

$$= \frac{1}{M} \sum_{1 \le k \le M} \left(\frac{1}{a} \log E \left(\sum_{\alpha} w_{\alpha,k} \mathrm{ch}(h + z_{k,\alpha}) \right)^a \right.$$

$$\left. - \frac{1}{a} \log E \left(\sum_{\alpha} w_{\alpha,k} \exp y_{k,\alpha} \right)^a \right) \le W^+(1)$$

because in each term of the sum we can replace $w_{\alpha,k}$ by $w_{\alpha,k} / \sum_\gamma w_{\gamma,k}$ and make the change of density $(\sum_\alpha w_{\alpha,k})^a / E(\sum_\alpha w_{\alpha,k})^a$. $\square$

Lemma 4.5. *For each q we have*

$$W^+(M) \ge B(q) := \log 2 + \frac{1}{2}(\xi(1) - \xi'(q))$$
$$+ \frac{1}{a} \log E \mathrm{ch}^a (h + z\sqrt{\xi'(q)}) - \frac{1}{2}(a-1)\theta(q). \qquad (4.22)$$

Proof: Let us take $A = \{1, \dots, R\}$, $q_{\alpha\beta} = 1$ if $\alpha = \beta$ and $q_{\alpha\beta} = q$ if $\alpha \ne \beta$. Consider independent standard normal r.v. $z, y, x_\alpha, x'_\alpha$ and take $z_\alpha = z\sqrt{\xi'(q)} + x_\alpha \sqrt{\xi'(1) - \xi'(q)}$ and $y_\alpha = y\sqrt{\theta(q)} + x'_\alpha \sqrt{\theta(1) - \theta(q)}$. It is simple to see that if $w_\alpha = R^{-1}$ for each $\alpha \in A$, this choice yields (4.22) as $R \to \infty$. Indeed the law of large numbers shows that for large R

$$\sum w_\alpha \exp \sqrt{M} y_\alpha \simeq \exp \frac{M}{2}(\theta(1) - \theta(q)) \exp \sqrt{M} y \sqrt{\theta(q)}$$

and

$$\sum_\alpha w_\alpha \prod_{j \le M} \mathrm{ch}(h + z_{j,\alpha}) \simeq \exp \frac{M}{2}(\xi'(1) - \xi'(q)) \prod_{j \le M} \mathrm{ch}(h + z_j \sqrt{\xi'(q)}).$$

□

For reasons that will become apparent later, the case $a \ge 1$ should be significantly simpler than the case $a < 1$. The previous two results are related to the following conjecture.

Conjecture 4.6. *If $a \ge 1$ for each M we have*

$$W^+(M) = W^+(1) = \sup_q B(q). \tag{4.23}$$

We will later prove (in a very different manner) that we have

$$\lim_{n \to \infty} \frac{U_N}{N} = \sup_q B(q). \tag{4.24}$$

It is plausible that, using the arguments of Ref. 5 one can deduce (4.23). This approach however is not interesting. What would be of interest would be to develop a "direct" proof of (4.23).

Another situation that one should meditate is the case $a = 0$. In that case

$$W^-(1) = \inf \left(E \log \sum w_\alpha \mathrm{ch}(h + z_\alpha) - E \log \left(\sum w_\alpha \exp y_\alpha \right) \right)$$

where this expectation is in $w_\alpha, z_\alpha, y_\alpha$. Thus, we also have

$$W^-(1) = \inf \left(E \log \sum w_\alpha \mathrm{ch}(h + z_\alpha) - E \log \left(\sum w_\alpha \exp y_\alpha \right) \right)$$

where now (w_α) are non random weights, and where the infimum is also on these weights.

5. POISSON DIRICHLET CASCADES

What are the weights considered in Sec. 4? In this section we construct some natural families of weights, that are believed to be universal.

We consider an integer k and $0 < m_1 < \cdots < m_k < 1$. Consider a non-increasing rearrangement $(u_i)_{i \ge 1}$ of the realization of a Poisson point process of intensity measure $x^{-m_1 - 1} dx$. And, for each l, $2 \le l \le k$, and each integers

$n_1, \ldots, n_{l-1}$ consider the non-increasing rearrangement $(u_{n_1,\ldots,n_{l-1},i})_{i\geq 1}$ of the realization of a Poisson point process with intensity measure $x^{-m_l-1}dx$. All these are independent. For a sequence $s = (s_1, \ldots, s_k)$ in $\mathbb{N}^{*k}$, and $l \leq k$, we write $u_{s|l} = u_{s_1,\ldots,s_l}$; and finally we define

$$v_s = u_{s|1} u_{s|2} \cdots u_s. \tag{5.1}$$

We recall the following:

Lemma 5.1. *Consider a r.v. $Y \geq 0$ with $EY^m \leq \infty$, and i.i.d. copies $(Y_i)_{i\geq 1}$ that are independent of a non-decreasing rearrangement (u_i) of a Poisson point process of intensity measure $x^{-m-1}dx$. Then the non-increasing rearrangement of the sequence $(u_i Y_i)$ has the same distribution as the sequence $((EY^m)^{1/m} u_i)$.*

Proof: See e.g., Ref. 12, p. 481.

Lemma 5.2. *Consider a sequence (v_s) as in (5.1). Then whenever $a < m_1$ we have $E(\sum_s v_s)^a < 1$, where the sum is over all choices of the sequence s.*

Proof: The proof is by induction over k. For $k = 1$ the lemma is true by Lemma 2.4. By induction hypothesis we have $EY_i^{m_1} < \infty$ where $Y_i = \sum_{s;s_1=i} u_{s|2} \ldots u_s$, that is, the sum is over all sequences s with $s_1 = i$. The r.v. $(Y_i)_{i\geq 1}$ are i.i.d. and independent from the r.v. u_i, and $\sum_v v_s = \sum_i u_i Y_i$, so, since $a < m_1$, we conclude using Lemmas 5.1 and 2.4. □

Lemma 5.3. *Consider a function $f(x_1, \ldots, x_k)$, and assume $f \geq 0$. Consider independent r.v. $X^1, \ldots, X^k$, and independent copies (X_i^1) of X^1; and for $n_1, \ldots, n_{l-1}$ $(2 \leq l \leq k)$, consider independent copies $(X^l_{n_1,\ldots,n_{l-1},i})$ of X^l. For $s \in \mathbb{N}^{*k}$ let*

$$f_s = f\big(X^1_{s_1}, X^2_{s_1,s_2}, \ldots, X^k_{s_1,\ldots,s_k}\big).$$

Then the sequence $(v_s f_s)_s$ can be rearranged to have the same distribution as the sequence (bv_s) where

$$b = (E_1(\ldots(E_{k-1}(E_k f^{m_k}(X^1, \ldots, X^k))^{m_{k-1}/m_k})^{m_{k-2}/m_{k-1}})\ldots))^{1/m_1}, \tag{5.2}$$

and where E_l means expectation in X^l only.

The expression "the sequence $(v_s f_s)$ can be rearranged" means that there is a (random) permutation ϱ such that the sequence $(v_{\varrho(s)} f_{\varrho(s)})_s$ has the same distribution as the sequence $(v_s f_s)$.

Proof: It goes again by induction over k. For $k = 1$, this is a consequence of Lemma 5.1. For the induction hypothesis from $k - 1$ to k, we observe that for each i, the sequence $(v_s f(X_i, X_{s|2}, \ldots, X_s))_{\{s;s_1=i\}}$ where s varies over the sequences

s for which $s_1 = i$, can be rearranged to have the distribution of the sequence $(v_s b_i)_{\{s; s_1 = i\}}$ where

$$b_i = \big(E_2\big(\dots\big(E_k f^{m_k}\big(X_i^1, X^2, \dots, X^k\big)\big)^{m_{k-1}/m_k} \dots\big)\big)^{1/m_2},$$

i.e. the distribution of the sequence

$$(u_i b_i u_{i,i_2} u_{i,i_2,i_3} \dots u_{i,i_2,\dots,i_k})_{i_2,\dots,i_k}.$$

The sequence $(u_i b_i)$ can be rearranged to the distributed like bu_i (and is independent of $(u_{i,i_2} u_{i,i_2,i_3} \dots u_{i,i_2,\dots,i_k})_{i_2,\dots,i_k}$). □

Theorem 5.4. *Consider the sequence $(v_s)_{s \in \mathbb{N}^{*k}}$ as above, and the weights $w_s = v_s / \sum_{s'} v_{s'}$. Let us denote by E' expectation after one has made a change of density $\left(\sum v_s\right)^a / E\left(\sum v_s\right)^a$. Then for a function $f(x_1, \dots, x_k) \geq 0$, if b is as in (5.2) and if $a < m_1$ we have*

$$E' \left(\sum w_s f\big(X_{s_1}^1, X_{s_1,s_2}^2, \dots, X_s^k\big) \right)^a = b^a.$$

Proof: Let us first note that $\sum v_s$ is finite a.s. since $E(\sum v_s)^a < \infty$ by Lemma 5.2. Also, by definition of w_s,

$$E'\big(\sum w_s f\big(X_{s_1}^1, X_{s_1,s_2}^2, \dots, X_s^k\big)\big)^a$$
$$= \frac{1}{E(\sum v_s)^a} E\big(\sum v_s f\big(X_{s_1}^1, \dots, X^{k_s}\big)\big)^a,$$

and the result follows by Lemma 5.3. □

We now explain a scheme to construct structures as in Sec. 4. Consider $a < m_1 < m_2 < \cdots < m_k < 1, m_1 > 0$. (It is possible that $a < 0$.) Consider $0 = q_0 < q_1 < \cdots < q_k < q_{k+1} = 1$, and for $s \in (\mathbb{N}^*)^k (= A)$ consider

$$z_s = z\sqrt{\xi'(q_1)} + z_{s_1}\sqrt{\xi'(q_2) - \xi'(q_1)} + \cdots + z_{s_1 \dots s_k}\sqrt{\xi'(1) - \xi'(q_k)}$$
$$y_s = z\sqrt{\theta(q_1)} + z_{s_1}\sqrt{\theta(q_2) - \theta(q_1)} + \cdots + z_{s_1 \dots s_k}\sqrt{\theta(1) - \theta(q_k)},$$

where $z, z_{s_1}, \dots$ are independent standard normal. We compute

$$\log 2 + \frac{1}{a} \log E' \left(\sum_s w_s \operatorname{ch}(h + z_s) \right)^a - \frac{1}{a} \log E' \left(\sum_s w_s \exp y_s \right)^a. \quad (5.3)$$

To compute $E'(\sum_s w_s \exp y_s)^a$, we use Theorem 5.4 at given z. Denoting by E_l expectation in z_l, we consider

$$b_{k+1} = \exp(z\sqrt{\theta(q_1)} + z_1\sqrt{\theta(q_2) - \theta(q_1)} + \cdots + z_k\sqrt{\theta(1) - \theta(q_k)}),$$

and we define recursively $b_l = (E_l b_{l+1}^{m_l})^{1/m_l}$, so that

$$b_l = \exp(z\sqrt{\theta(q_1)} + \cdots + z_{l-1}\sqrt{\theta(q_l) - \theta(q_{l-1})})$$
$$\times \exp\left(\frac{1}{2}m_k(\theta(1) - \theta(q_k)) + \cdots + \frac{1}{2}m_l(\theta(q_{l+1}) - \theta(q_l))\right)$$

and

$$E'_{-}\left(\sum_s w_s \exp y_s\right)^a = b_1^a$$

where E'_{-} denotes E' at z given. Taking expectation in z, and setting $m_0 = a$, we find that

$$\frac{1}{a}\log E'\left(\sum_s w_s \exp y_s\right)^a = \frac{1}{2}\sum_{0\le l\le k} m_l(\theta(q_{l+1}) - \theta(q_l)).$$

It is because of this last expectation, in z, that the possibly negative value of a is permitted. In a similar manner we define

$$X_{k+1} = \log \mathrm{ch}(h + z\sqrt{\xi'(q_1)} + \cdots + z_k\sqrt{\xi'(1) - \xi'(q_k)}) \tag{5.4}$$

and recursively for $l \ge 0$, denoting by E_0 expectation in z,

$$X_l = \frac{1}{m_l}\log E_l \exp m_l X_{l+1}, \tag{5.5}$$

we find that

$$\frac{1}{a}\log E'\left(\sum_s w_s \mathrm{ch}(h + z_s)\right)^a = X_0 \tag{5.6}$$

and the quantity (5.3) is

$$\log 2 + X_0 - \frac{1}{2}\sum_{0\le l\le k} m_l(\theta(q_{l+1}) - \theta(q_l)). \tag{5.7}$$

It follows from Corollary 4.3 that for $a < 1$ this quantity is an upper bound for $W^-(1)$. The quantity (5.7) is the famous Parisi formula with corresponding "functional order parameter" the function $m(q)$ such that $m(q) = m_l$ for $q_l \le q < q_{l+1}$, $0 \le l \le k$. This function is a non-decreasing function $[0, 1] \to [a, 1]$. Since we have defined $m_0 = a$, and since $m_1 > 0$, when $a < 0$ this function is actually valued in $\{a\} \cup [0, 1]$. This rich structure when $a < 1$ contrasts with the case $a > 1$. In that case, the only interesting structure we can think of is that described in Lemma 4.5. This is natural in view of (4.24).

Conjecture 5.5. (The general Parisi conjecture) *With the notations of Sec. 4, if $a < 1$, we have $W^-(M) = W(1)$ for each M and this quantity is the infimum of the quantities (5.6) over all choices of k, $a = m_0 < m_1 < \cdots < m_k < 1$, $0 \leq q_1 < \cdots < q_k \leq q_{k+1} = 1$.*

The case $a < 0$ is entirely open. As we will explain in the next section, this case raises very interesting issues.

6. GUERRA'S BOUND AND GUERRA'S INVERSE BOUND

In this section we consider the Hamiltonian (3.2), and we assume that the function ξ of (3.4) is convex. Consider $a < 1$, and

$$a = m_0 < m_1 < \cdots < m_k < 1 \tag{6.1}$$

$$q_0 = 0 < q_1 < \cdots < q_k < q_{k+1} = 1. \tag{6.2}$$

In contrast with Sec. 5, we do NOT assume that $m_1 > 0$, so that one or more of the numbers m_l can be < 0. Let us define X_0 as in (5.6).

Theorem 6.1 (Guerra's bound). *For each choice of parameters as above, we have, with our usual notation*

$$\frac{1}{Na} \log EZ_N^a \leq \log 2 + X_0 - \frac{1}{2} \sum_{0 \leq l \leq k} m_l(\theta(q_{l+1}) - \theta(q_l)). \tag{6.3}$$

A striking consequence of Conjecture 5.5 is the purely analytical fact that when $a < 0$ the infimum in the right hand side of (6.3) is the same whether one allow $a = m_0 < m_1 < \cdots \leq 1$ or whether one requires $a = m_0 < 0 \leq m_1 < \cdots \leq 1$. In Sec. 10 we will prove that this is indeed the case for the spherical model; but we do not know if this is true for the SK model.

Proof of Guerra's bound: This proof requires only minor modifications from the case $a = 0$, as detailed in Ref. 13. To explain what these modifications are, let us consider

$$H_t(\boldsymbol{\sigma}) = \sqrt{t} H_N(\boldsymbol{\sigma}) + \sum_{i \leq N} \sigma_i \left(h + \sqrt{1-t} \sum_{0 \leq p \leq k} z_{i,p} \right),$$

where $z_{i,p}$ are independent standard Gaussian r.v. with $Ez_p^2 = \xi'(q_{p+1}) - \xi'(q_p)$. We define

$$F_{k+1,t} = \log \sum_{\boldsymbol{\sigma}} \exp H_t(\boldsymbol{\sigma})$$

and by recursion

$$F_{l,t} = \frac{1}{m_l} \log E_l \exp m_l F_{l+1,t},$$

where E_l denotes expectation in the r.v. $z_{i,p}$, $p \geq l$. Let us define

$$\varphi(t) = \frac{1}{N} F_{0,t} = \frac{1}{aN} \log E \exp a F_{1,t}.$$

For $0 \leq l \leq k$, let

$$W_l = \exp m_l (F_{l+1,t} - F_{l,t}),$$

so that in particular

$$W_0 = \exp a(F_{1,t} - F_{0,t}) = \frac{\exp a F_{1,t}}{E \exp a F_{1,t}}.$$

When $a = 0$, we have $W_0 = 1$, but this is not the case when $a \neq 0$. In the formula of Ref. 13, Sec. 3, we must replace the quantity $W_1 \cdots W_k$ by $W_0 W_1 \cdots W_k$, e.g. the formula (3.2) there becomes

$$\varphi'(t) = E\left(W_0 \cdots W_k \frac{\partial F_{k+1,t}}{\partial t}\right).$$

Also, we have, for all numbers c_l

$$\sum_{0 \leq l \leq k} m_l (c_{l+1} - c_l) = c_{k+1} - a c_0 + \sum_{1 \leq l \leq k} c_l (m_{l-1} - m_l), \tag{6.4}$$

and thus, if $T_l = \exp F_{l,t}(m_{l-1} - m_l)$ we have

$$\begin{aligned} W_0 \cdots W_k &= \exp\left(\sum_{0 \leq l \leq h} m_l (F_{l+1,t} - F_{l,t})\right) \\ &= \exp(-a F_{0,t}) T_1 \cdots T_k \exp F_{k+1,t}. \end{aligned}$$

The new term $\exp(-a F_{0,t})$ is a *number*, so that it does <u>not</u> interfere with the integration by parts. Also, since $\theta(0) = 0$, we still have

$$\theta(1) + \sum_{1 \leq l \leq k} (m_{l-1} - m_l)\theta(q_l) = \sum_{0 \leq l \leq k} m_l (\theta(q_{l+1}) - \theta(q_l))$$

as is used in the line above equation (3.17) of Ref. 13. No other changes are required. □

Let us also note the following:

Proposition 6.2 (Guerra's reverse bound). *Suppose now that* $a = m_0 \geq m_1 \geq \cdots \geq m_k = 1$*, and define* X_0 *as in (5.4), (5.5). Then*

$$\frac{1}{Na}\log EZ_N^a \geq \log 2 + X_0 - \frac{1}{2}m_l(\theta(q_{l+1}) - \theta(q_l)). \tag{6.5}$$

The proof is identical to the proof of Theorem 6.1. A consequence of (4.24) is the purely analytic fact that the lower bound is in fact the same when one takes $k = 1$ and $m_k = 1$. This does not seem obvious at all a priori. That is indeed the case follows also from a deep recent result of D. Panchenko, who proves[10] that the right hand side of (6.5) is separately convex in each of the variables m_l, and therefore that (at $q_0, q_1, \ldots$ given) the maximum of this right hand side can be obtained only when all numbers m_l are equal to either a or 1. (Therefore (6.5) does not contradict (4.22).)

7. THE CASE $0 \leq a \leq 1$

In this section we assume that the function ξ is convex. Intuitively, it is natural to expect that this case is easier than the case $a < 0$, because in this case no work is required to show that Guerra's bound of Sec. 6 coincides with the natural bounds of Sec. 5. The case $a = 0$ is proved in Ref. 13, and the case $0 < a < 1$ is very similar and, interestingly, is somewhat easier. This is because in Ref. 13 special arguments were required to deal with the fact that $m_0 = 0$, and this does not occur when $m_0 = a > 0$. Thus there is no need for Propositions 5.3 and 5.4 of Ref. 13. On the other hand, we need a substitute for Lemma 2.6 of Ref. 13. It turns out that it suffices to make a small change at the very end of the proof of this lemma to obtain the required substitute. Keeping the notations of Ref. 13, we now have $n_0 = m_0/2 = a/2$. Let us recall that $J_{k+1} = 2\log Z_N$, that

$$J_{k+1,u} = \log \sum_{R_{1,2}u} \exp\left(H_N(\boldsymbol{\sigma}^1) + H_N(\boldsymbol{\sigma}^2) + h\sum_{i\leq N}\left(\sigma_i^1 + \sigma_i^2\right)\right),$$

that the numbers J_l are defined recursively by $J_l = \frac{1}{n_l}\log E_l \exp n_l J_{l+1}$ (and similarly for the numbers $J_{l,u}$), and that $V_l = \exp n_l(J_{l+1} - J_l)$.

Lemma 7.1. *If* $J_{0,u} < J_0 - N\varepsilon$*, then for some constant* K' *independent of* N *or* t *we have*

$$E(V_0V_1\cdots V_k\langle 1_{\{R_{1,2}=u\}}\rangle) \leq K'\exp\left(-\frac{N}{K'}\right). \tag{7.1}$$

Proof: Let us denote by A the left-hand side of (7.1), so

$$A = E\left(\prod_{0\leq l\leq k}\exp n_l(J_{l+1} - J_l)\frac{\exp J_{k+1,u}}{\exp J_{k+1}}\right).$$

Since $J_{k+1,u} \leq J_{k+1}$ and $n_k \leq 1$, we have

$$\exp(J_{k+1,u} - J_{k+1}) \leq \exp n_k(J_{k+1,u} - J_{k+1})$$

so

$$\begin{aligned} A &\leq E\left(\prod_{0\leq l\leq k-1} \exp n_l(J_{l+1} - J_l) \exp n_k(J_{k+1,u} - J_{k+1})\right) \\ &= E\left(\prod_{0\leq l\leq k-1} \exp n_l(J_{l+1} - J_l) \exp n_k(J_{k,u} - J_k)\right) \end{aligned}$$

because $E_k \exp n_k J_{k+1,u} = \exp n_k J_{k,u}$ and all of the other terms are independent of the r.v. $z_{i,k}$. Continuing in this manner, we see that

$$A \leq \exp n_0(J_{0,u} - J_0) \leq \exp(-N n_0 \varepsilon).$$

□

No other change to the arguments of Ref. 13 are needed. One difficulty we cannot pass in the case $a < 0$ is to find a substitute to the previous lemma. To explain the difficulty let us set $b = -a > 0$, and let us consider the simpler problem of the control of the "high-temperature region." Let us first observe that controlling from above

$$\frac{1}{a} \log E Z^a = -\frac{1}{b} \log E \frac{1}{Z^b}$$

means controlling EZ^{-b} from below.

Given u (and N) let

$$B(u) = \sum_{R_{1,2}=u} \exp\left(H_N(\boldsymbol{\sigma}^1) + H_N(\boldsymbol{\sigma}^2) + h\sum_{i\leq N} \left(\sigma_i^1 + \sigma_i^2\right)\right).$$

We can conceive that one could adapt the arguments of Ref. 12 , pp. 154–155 in order to prove that if $u \neq q$ (where q is the value of (3.6)), then for some number ε independent of N one has

$$E\frac{1}{B(u)^{b/2}} \geq e^{\varepsilon N} E\frac{1}{Z^b}, \tag{7.2}$$

or, equivalently,

$$E\left(\left(\frac{Z_N^2}{B(u)}\right)^{b/2} \frac{1}{Z_N^b}\right) \gg E\frac{1}{Z_N^b}. \tag{7.3}$$

But this is not the information one needs. The information one needs to conduct the proof is that

$$E\left(\frac{B(u)}{Z_N^2}\frac{1}{Z_N^b}\right) \ll E\frac{1}{Z_N^b}, \tag{7.4}$$

which we do not see how to deduce from (7.3) or to prove directly.

8. THE CASE $1 \le a \le 2$

In this section we prove the following:

Theorem 8.1. *Assume that* $1 \le a \le 2$. *Then, for the Hamiltonian (3.2), and if the function* ξ *is convex with* $\xi'' > 0$, *we have*

$$\lim_{N\to\infty} \frac{1}{Na} \log E Z_N^a = \sup_q \left(\log 2 + \frac{1}{2}(\xi(1) - \xi'(q) + (1-a)\theta(q)) + \frac{1}{a} \log E\mathrm{ch}^a(z\sqrt{\xi'(q)} + h)\right). \tag{8.1}$$

Of course here z is standard normal and $\theta(q) = q\xi'(q) - \xi(q)$. In the next section we will present arguments that show that in fact (8.1) holds for any $a \ge 1$. Although this sounds funny at first, our main motivation for presenting the proof of Theorem 8.1 is that we do not see how to extend it to the case $a \ge 2$. We feel that the difficulty that arises when attempting to do this could be of a rather fundamental nature. It resembles the difficulty that arises when one tries to solve the ultrametricity problem or the chaos problem.[15]

Let us fix once and for all a value of q that achieves the supremum in the right-hand side of (8.1). Let us define the interpolating Hamiltonian

$$H_t(\boldsymbol{\sigma}) = \sqrt{t}H_N(\boldsymbol{\sigma}) + \sqrt{1-t}\sum_{i\le N} z_i\sqrt{\xi'(q)}\sigma_i + h\sum_{i\le N}\sigma_i, \tag{8.2}$$

where of course the r.v. z_i are i.i.d. standard normal. Let us define

$$\psi(t) = \log 2 + \frac{t}{2}(\xi(1) - \xi'(q) + (1-a)\theta(q)) + \frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{\xi'(q)} + h) \tag{8.3}$$

and

$$\Psi(t,u) = \frac{1}{Na}\log E\left(\sum_{R_{1,2}u} \exp(H_t(\boldsymbol{\sigma}^1) + H_t(\boldsymbol{\sigma}^2))\right)^{a/2}. \tag{8.4}$$

The key to Theorem 8.1 is the following:

Proposition 8.2. *If $t < 1$ we have*

$$\Psi(t,u) \le \psi(t) - K(1-t)(u-q)^2, \tag{8.5}$$

where K does not depend on t or N.

Proposition 8.3. *Assume that for some $\varepsilon \ge 0$ we have*

$$\Psi(t,u) \le \frac{1}{Na}\log E Z_t^a - \varepsilon. \tag{8.6}$$

Then we have

$$E\left(\frac{Z_t^a}{E Z_t^a}\langle 1_{\{R_{1,2u}\}}\rangle_t\right) \le K\exp\left(-\frac{N}{K}\right), \tag{8.7}$$

where K is independent of N.

Of course $\langle\cdot\rangle_t$ denotes an average for the Gibbs measure with Hamiltonian (8.2).

Proof: Let

$$A = \sum_{R_{1,2}u}\exp(H_t(\boldsymbol{\sigma}^1) + H_t(\boldsymbol{\sigma}^2))$$

so that, since $A \le Z_t^2$ and $a/2 < 1$

$$E\big(Z_t^a\langle 1_{\{R_{1,2}=u\}}\rangle_t\big) = E\big(Z_t^{a-2}A\big) \le E\big(Z_t^{a-2}Z_t^{2(1-a/2)}A^{a/2}\big) = E(A^{a/2})$$

and

$$E\left(\frac{Z_t^a}{E Z_t^a}\langle 1_{\{R_{1,2}=u\}}\rangle_t\right) \le \frac{E A^{a/2}}{E Z_t^a} \le \exp aN\left(\Psi(t,u) - \frac{1}{aN}\log E Z_t^a\right).$$

□

Once we know Propositions 8.2 and 8.3, we proceed as in Ref. 13, proof of Theorem 2.2 to obtain Theorem 8.1 through differential inequalities. We turn to the proof of Proposition 8.2.

Lemma 8.4. *We have*

$$\Psi(t,u) \le \psi(t,u) := \log 2 + \frac{t}{2}(\xi(1) - \xi'(u) + (1-a)\theta(u)) + \frac{1}{a}\log E\,\mathrm{ch}^a(z\sqrt{(1-t)\xi'(q) + t\xi'(u)} + h). \tag{8.8}$$

Proof: Consider new independent standard Gaussian r.v. $(y_i)_{i\le N}$, and the interpolating Hamiltonian

$$H_v(\boldsymbol{\sigma}) = \sqrt{vt}H_N(\boldsymbol{\sigma}) + \sqrt{(1-v)t}\sum_{i\le N} y_i\sqrt{\xi'(u)}\sigma_i + \sqrt{1-t}\sum_{i\le N} z_i\sqrt{\xi'(q)}\sigma_i + h\sum_{i\le N}\sigma_i.$$

Let

$$f(v) = \frac{1}{aN}\log\left(EZ_v^{a/2}\right),$$

where

$$Z_v = \sum_{R_{1,2}=u}\exp(H_v(\boldsymbol{\sigma}^1) + H_v(\boldsymbol{\sigma}^2)),$$

so that it is very simple to see that

$$f(0) = \log 2 + \frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{(1-t)\xi'(q) + t\xi'(u)} + h).$$

Now

$$f'(v) = \frac{1}{2N}\frac{E\left(\frac{dZ_v}{dv}Z_v^{a/2-1}\right)}{EZ_v^{a/2}},$$

and computation using integration by parts yields

$$2f'(v) = t\Bigg(\xi(1) - \xi'(u) + \xi(u) - u\xi'(u) + \frac{1}{2}\left(\frac{a}{2}-1\right) \times \sum_{l,l'} E\left(\frac{EZ_v^{a/2}}{EZ_v^{a/2}}\langle\xi(R(\boldsymbol{\sigma}^l,\boldsymbol{\tau}^{l'})) - R(\boldsymbol{\sigma}^l,\boldsymbol{\tau}^{l'})\xi'(u)\rangle_v\right)\Bigg).$$

Here $l, l' = 1, 2$, $(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2)$ is a replica of the system $(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$, $\langle\cdot\rangle_v$ is a certain Gibbs average and $R(\boldsymbol{\sigma},\boldsymbol{\tau}) = N^{-1}\sum_{i\le N}\sigma_i\tau_i$. Using that $a/2 - 1 < 0$ and

$$\xi(R(\boldsymbol{\sigma}^l,\boldsymbol{\tau}^{l'})) - R(\boldsymbol{\sigma}^l,\boldsymbol{\tau}^l)\xi'(u) \ge -\theta(u)$$

we see that

$$\begin{aligned}2f'(v) &\le t\left(\xi(1) - \xi'(u) - \theta(u) + 2\left(1 - \frac{a}{2}\right)\theta(u)\right)\\ &= t(\xi(1) - \xi'(u) + (1-a)\theta(u)),\end{aligned}$$

and this proves (8.8). □

To conclude the proof of Theorem 8.1, we show the following:

Lemma 8.5. *We have*

$$\psi(t,u) \le \psi(t) - K(1-t)(u-q)^2. \tag{8.9}$$

Proof: Let us write

$$B(u) = \xi(1) - \xi'(u) + (1-a)\theta(u).$$

Let us define the number v by

$$\xi'(v) = (1-t)\xi'(q) + t\xi'(u), \tag{8.10}$$

so that

$$\psi(t,u) = \psi(1,v) + tB(u) - B(v). \tag{8.11}$$

Now, by the choice of q we have $\psi(1,v) \le \psi(1,q)$, and

$$\psi(1,q) = \psi(t) + (1-t)B(q), \tag{8.12}$$

so that, using (8.10) and the definition of B,

$$\begin{aligned}\psi(t,u) &\le \psi(t) + tB(u) + (1-t)B(q) - B(v)\\ &= \psi(t) + (a-1)(\theta(v) - t\theta(u) - (1-t)\theta(q)).\end{aligned} \tag{8.13}$$

The function $U(x)$ given by $U(\xi'(x)) = \theta(x)$ satisfies $U'(\xi'(x)) = x$ since $\theta'(x) = x\xi''(x)$. It follows that U'' is bounded below by a positive number on the interval $[\xi'(-1), \xi'(1)]$, and thus, using (8.10),

$$U(\xi'(v)) \le tU(\xi'(u)) + (1-t)U(\xi'(q)) - \frac{t(1-t)}{K}(\xi'(q) - \xi'(u))^2$$

and thus, since we assume $\xi'' > 0$,

$$\theta(v) - t\theta(u) - (1-t)\theta(q) \le -\frac{t(1-t)}{K}(q-u)^2,$$

where K depends on the function ξ only, so by (8.13) we have

$$\psi(t,u) \le \psi(t) - \frac{t(1-t)}{K}(q-u)^2. \tag{8.14}$$

To prove (8.8) it now suffices to prove that there is $t_0 > 0$ such that (8.9) holds for $t < t_0$. But it is easily seen that for t small the map $u \mapsto \psi(t,u)$ has a unique maximum at $u = q$ (and that the second derivative at $u = q$ stays away from zero). □

Of course, it is probably easy to remove the restriction that $\xi'' > 0$. For example our very argument shows that if $h \ne 0$ it suffices to assume that $\xi''(q) > 0$ for $q \ne 0$.

How could one extend the proof of Theorem 8.1 to the case $a > 2$? To generalize the previous approach, one would need to extend (8.5) to the case where

$$\Psi(t, u) = \frac{1}{Na} \log E \left(\sum_{R_{1,2}=u} \exp \sum_{l \le n} H_t(\boldsymbol{\sigma}^l) \right)^{a/n} \tag{8.15}$$

where n is an integer $\ge a$. When using interpolation as in Lemma 8.4, to avoid the terms with the wrong sign, we should first try to bound a quantity such as

$$\Psi(t) = \frac{1}{Na} \log E \left(\sum_{R_{l,l'}=u_{l,l'}} \exp \sum_{l \le n} H_t(\boldsymbol{\sigma}^l) \right)^{a/n} .$$

That is, the summation is restricted to the configurations $\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n$ with $R_{l,l'} = N^{-1} \sum_{i \le N} \sigma_i^l \sigma_i^{l'} = u_{l,l'}$ for $1 \le l < l' \le n$. (So $u_{l,l} = 1$.) Let us now consider numbers $q_{l,l'}$ for $l, l' \le n$, and centered Gaussian r.v. y_l with $E y_l y_{l'} = \xi'(q_{l,l'})$. Let us consider independent copies $(y_{l,i})_{i \le N}$ of the r.v. $(y_l)_{l \le n}$ and, for $0 \le v \le 1$, the Hamiltonian

$$\begin{aligned} H_v(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) = \sum_{l \le n} \sqrt{vt} H_N(\boldsymbol{\sigma}^l) + \sqrt{(1-v)t} \sum_{l \le n} \sum_{i \le N} y_{l,i} \sigma_i^l \\ + \sqrt{1-t} \sum_{l \le n} \sum_{i \le N} z_i \sqrt{\xi'(q)} \sigma_i^l + h \sum_{l \le n} \sum_{i \le N} \sigma_i^l , \end{aligned}$$

and let us define

$$Z_v = \sum_{R_{l,l'}=u_{l,l'}} \exp H_v(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n).$$

Let us consider $f(v)$ given by

$$f(v) = \frac{1}{aN} \log E Z_v^{a/n}.$$

If we copy the proof of Lemma 8.4 we find that

$$f'(v) \le \frac{t}{2n} \sum_{l,l' \le n} \left(\xi(u_{l,l'}) - u_{l,l'} \xi'(q_{l,l'}) + \left(1 - \frac{a}{n}\right) \theta(q_{l,l'}) \right).$$

Now, given numbers $\lambda_{l,l'}$, for $l < l'$

$$\begin{aligned} f(0) &= \frac{1}{aN} \log E \left(\sum_{R_{l,l'}=u_{l,l'}} \exp \sum_{i \le N} \sum_{l \le n} \sigma_i^l (\sqrt{t} y_{l,i} + \sqrt{1-t} z_i \sqrt{\xi'(q)} + h) \right)^{a/n} \\ &= \frac{1}{aN} \log E \left(\sum_{R_{l,l'}=u_{l,l'}} \exp \sum_{i \le N} A_i \right)^a - \sum_{l<l'} \lambda_{l,l'} u_{l,l'}, \end{aligned}$$

where

$$A_i = \sum_{l\le n} \sigma_i^l(\sqrt{t}y_{l,i} + \sqrt{1-t}z_i\sqrt{\xi'(q)} + h) + \sum_{l<l'} \sigma_i^l\sigma_i^{l'}\lambda_{l,l'}.$$

By independence

$$E\left(\sum_{R_{l,l'}=u_{l,l'}} \exp\sum_{i\le N} A_i\right)^{a/n} \le E\left(\sum_{\sigma^1,\dots,\sigma^l} \exp\sum_{i\le N} A_i\right)^{a/n} = \prod_{i\le N} E\left(\sum \exp A_i\right)^{a/n}$$

$$= \left(E\left(\sum \exp\left(\sum_{l\le n} \varepsilon_l(\sqrt{t}y_l + \sqrt{1-t}z\sqrt{\xi'(q)} + h) + \sum_{l<l'} \varepsilon_l\varepsilon_{l'}\lambda_{l,l'}\right)\right)^{a/n}\right)^N,$$

where the summation is over all values $\varepsilon_l, \varepsilon_{l'} = \pm 1$.

Thus, we have shown that

$$\Psi(t) \le \frac{t}{2n}\sum_{l,l'\le n}\left(\xi(u_{l,l'}) - u_{l,l'}\xi'(q_{l,l'}) + \left(1-\frac{a}{n}\right)\theta(q_{l,l'})\right)$$
$$+\frac{1}{a}\log E\left(\sum\exp\left(\sum_{l\le n}\varepsilon_l(\sqrt{t}y_l + \sqrt{1-t}z\sqrt{\xi'(q)} + h)\right.\right.$$
$$\left.\left.+\sum_{l<l'}\varepsilon_l\varepsilon_{l'}\lambda_{l,l'}\right)\right)^{a/n} - \sum_{l<l'}\lambda_{l,l'}u_{l,l'}. \tag{8.16}$$

The problem is as follows. Assuming $u_{1,2} \neq q$, prove that one can find the numbers $q_{l,l'}$ and the numbers $\lambda_{l,l'}$ such that the right-hand side of (8.16) is $< \psi(t)$. Of course, the reader might think that the reason why this looks like a difficult problem is that we do not use the "correct" bound for $\Psi(t)$. But there is some evidence that this bound is correct in the case $a = n$. In this case, we have

$$E\exp\sum_{l\le n}\varepsilon_l\sqrt{t}y_l = \exp\left(\frac{1}{2}\sum_{l\le n}t\xi'(q_{l,l}) + \sum_{l<l'}\varepsilon_l\varepsilon_{l'}t\xi'(q_{l,l'})\right)$$

and, since

$$\sum_{l,l'} -u_{l,l'}\xi'(q_{l,l'}) = -\sum_{l\le n}\xi'(q_{l,l}) - 2\sum_{l<l'}u_{l,l'}\xi'(q_{l,l'}),$$

the right-hand side of (8.16) is

$$\frac{t}{2n}\sum_{l,l'}\xi(u_{l,l'}) + \frac{1}{n}\log E\left(\sum_{\varepsilon_l}\exp\left(\sum_{l\le n}\varepsilon_l(\sqrt{1-t}z\sqrt{\xi'(q)} + h)\right.\right.$$
$$\left.\left.+\sum_{l<l'}\varepsilon_l\varepsilon_{l'}\lambda'_{l,l'}\right)\right) - \frac{1}{n}\sum_{l<l'}\lambda'_{l,l'}u_{l,l'}, \tag{8.17}$$

where $\lambda'_{l,l'} = \lambda_{l,l'} + t\xi'(q_{l,l'})$. It seems likely that the infimum over all choices of $\lambda'_{l,l'}$ of the quantity (8.17) is

$$\frac{t}{2n}\sum_{l<l'}\xi(u_{l,l'}) + \frac{1}{nN}\log E\left(\sum_{R_{l,l'}=u_{l,l'}}\exp\left(\sum_{l\le n}\sum_{i\le N}\sigma_i^l(\sqrt{1-t}z_i\sqrt{\xi'(q)}+h)\right)\right)$$

$$= \frac{1}{nN}\log E\left(\sum_{R_{l,l'}u_{l,l'}}\exp\left(\sum_{l\le n}H_t(\boldsymbol{\sigma}^l)\right)\right) = \Psi(t),$$

confirming that the bound (8.7) is "correct."

Before controlling the right-hand side of (8.16) a more urgent problem is as follows.

Problem 8.6. *It is true that if $u_{1,2} \neq q$, the infimum of the quantity (8.17) over all choices of the numbers $(\lambda_{l,l'})$ is $< \psi(t)$?*

9. THE CASE $a > 1$ THROUGH THE GHIRLANDA-GUERRA IDENTITIES

In this section we still assume that the function ξ is convex and we compute the limit (1.2) for $a > 1$ for the Hamiltonian (3.2). While the arguments are not complicated, the author finds that it is far from being really clear why they work. These arguments are a kind of variation on the Ghirlanda-Guerra identities,[7] see e.g. Ref. 12, Chapter 6. (The author does not really understand either why these inequalities hold, and progress on this question would be most welcomed.)

The first part of the argument is very general. We consider a random Hamiltonian $H_N(\boldsymbol{\sigma})$, and an i.i.d. standard Gaussian sequence $(h_i)_{i\le N}$ that is independent of the randomness of H_N. We consider a sequence (c_N), and, for $u \in \mathbb{R}$, the Hamiltonian

$$H_{N,u}(\boldsymbol{\sigma}) = H_N(\boldsymbol{\sigma}) + uc_N\sum_{i\le N}h_i\sigma_i. \tag{9.1}$$

Many useful choices of the sequence c_N are possible, e.g. $c_N = N^{-1/4}$. We write

$$Z_{N,u} = \sum_{\boldsymbol{\sigma}}\exp H_{N,u}(\boldsymbol{\sigma}), \tag{9.2}$$

$$\varphi(u) = \frac{1}{Na}\log EZ^a_{N,u}. \tag{9.3}$$

We write

$$B = B(\boldsymbol{\sigma}) = \sum_{i\le N}h_i\sigma_i,$$

and it is straightforward that

$$\varphi'(u) = \frac{c_N}{N} E\left(\frac{\sum_{\boldsymbol{\sigma}} B(\boldsymbol{\sigma}) \exp H_{N,u}(\boldsymbol{\sigma}) Z_{N,u}^{a-1}}{E Z_{N,u}^{a}}\right)$$
$$= \frac{c_N}{N} E'\langle B \rangle, \tag{9.4}$$

where E' denotes expectation after change of density by $Z_{N,u}^{a}/E Z_{N,u}^{a}$, and where $\langle \cdot \rangle$ denotes an average for the Gibbs measure with Hamiltonian (9.1) (so the value of u is implicit in this notation).

Lemma 9.1. *We have*

$$E'\langle B \rangle N u c_N (1 + (a-1) E'\langle R_{1,2} \rangle). \tag{9.5}$$

Proof: This is straightforward integration by parts, based on the fact that

$$E B(\boldsymbol{\sigma}^1) B(\boldsymbol{\sigma}^2) = N R_{1,2} = \sum_{i \le N} \sigma_i^1 \sigma_i^2.$$

□

Combining with (9.4) we see that

$$\varphi'(0) = 0, \quad \varphi'(1) \le a c_N^2. \tag{9.6}$$

Lemma 9.2. *We have*

$$\int_0^1 \left(E'\left\langle \frac{B^2}{N^2} \right\rangle - \left(E'\left\langle \frac{B}{N} \right\rangle \right)^2 \right) du \le \frac{a}{N}. \tag{9.7}$$

Proof: Starting from (9.4) it is straightforward to see that

$$\varphi''(u) = \frac{c_N^2}{N} (E'\langle B^2 \rangle + (a-1) E'\langle B \rangle^2 - a (E'\langle B \rangle)^2). \tag{9.8}$$

Since $a > 1$ and $E'\langle B \rangle^2 \ge (E'\langle B \rangle)^2$, we have

$$\varphi''(u) \ge \frac{c_N^2}{N} (E'\langle B^2 \rangle - (E'\langle B \rangle)^2). \tag{9.9}$$

Moreover we have $\int_0^1 \varphi''(u)\, du = \varphi'(1) - \varphi'(0) \le a c_N^2$ from (9.6). The result follows. □

Lemma 9.3. *For some number $K(a)$ depending on a only we have*

$$\int_0^1 u(E'\langle R_{1,2}^2\rangle - (E'\langle R_{1,2}\rangle)^2)\,du \leq \frac{K(a)}{c_N\sqrt{N}}. \tag{9.10}$$

This is the key point. For the typical value of u, the overlap $R_{1,2}$ is nearly constant.

Proof: We have, since $|R_{1,2}| \leq 1$

$$\begin{aligned} E'\left\langle \frac{B(\boldsymbol{\sigma}^1)}{N} R_{1,2} \right\rangle - E'\left\langle \frac{B}{N} \right\rangle E'\langle R_{1,2}\rangle &\leq E'\left\langle \left| \frac{B}{N} - E'\left\langle \frac{B}{N} \right\rangle \right| \right\rangle \\ &\leq \left(E'\left\langle \left(\frac{B}{N} - E'\left\langle \frac{B}{N} \right\rangle \right)^2 \right\rangle \right)^{1/2} \\ &= \left(E'\left\langle \frac{B^2}{N^2} \right\rangle - \left(E'\left\langle \frac{B}{N} \right\rangle \right)^2 \right)^{1/2}. \end{aligned} \tag{9.11}$$

By integration by parts as in (9.5) we get that

$$E'\left\langle \frac{B(\boldsymbol{\sigma}^1)}{N} R_{1,2} \right\rangle uc_N \left(E'\langle R_{1,2}\rangle + E'\left\langle R_{1,2}^2 \right\rangle + (a-2)E'\langle R_{1,2}R_{1,3}\rangle \right) \tag{9.12}$$

and combining with (9.5) we get

$$E'\left\langle \frac{B(\boldsymbol{\sigma}^1)}{N} R_{1,2} \right\rangle - E'\left\langle \frac{B}{N} \right\rangle E'\langle R_{1,2}\rangle = uc_N V \tag{9.13}$$

where

$$V = E'\left\langle R_{1,2}^2 \right\rangle + (1-a)(E'\langle R_{1,2}\rangle)^2 + (a-2)E'\langle R_{1,2}R_{1,3}\rangle.$$

We observe that $\langle R_{1,2}R_{1,3}\rangle \geq \langle R_{1,2}\rangle^2$. This is simply the inequality

$$\begin{aligned} \int f(x_1,x_2)f(x_1,x_3)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x_3) &= \int \left(\int f(x,y)\,d\mu(y) \right)^2 d\mu(x) \\ &\geq \left(\int f(x,y)\,d\mu(x)\,d\mu(y) \right)^2. \end{aligned}$$

Thus we have

$$E'\langle R_{1,2}^2\rangle \geq E'\langle R_{1,2}R_{1,3}\rangle \geq (E'\langle R_{1,2}\rangle)^2.$$

When $a \leq 2$, we use that

$$E'\langle R_{1,2}^2\rangle \geq (a-1)E'\langle R_{1,2}^2\rangle + (2-a)E'\langle R_{1,2}R_{1,3}\rangle$$

to get

$$V \geq (a-1)\big(E'\langle R_{1,2}^2\rangle - (E'\langle R_{1,2}\rangle)^2\big). \tag{9.14}$$

When $a \geq 2$, we use that $E'\langle R_{1,2}R_{1,3}\rangle \geq (E'\langle R_{1,2}\rangle)^2$ to get

$$V \geq E'\langle R_{1,2}^2\rangle - (E'\langle R_{1,2}\rangle)^2. \tag{9.15}$$

Combining (9.5) to (9.15) we get

$$uc_N \min(1, a-1)(E'\langle R_{1,2}^2\rangle - (E'\langle R_{1,2}\rangle)^2) \leq \left(E'\left\langle \frac{B^2}{N^2}\right\rangle - \left(E'\left\langle \frac{B}{N}\right\rangle\right)^2\right)^{1/2}.$$

Integration over $0 \leq u \leq 1$ and use of (9.7) conclude the proof. □

Let us now define

$$\mathrm{RS} = \sup_q \left(\log 2 + \frac{1}{2}(\xi(1) - \xi'(q)) + \frac{1}{a}\log E\operatorname{ch}^a(h + z\sqrt{\xi'(q)}) - \frac{1}{2}(a-1)\theta(q)\right)$$

where ξ is the function (3.4).

Theorem 9.4. *If the fucntion ξ is convex and if H_N denotes the Hamiltonian (3.2), we have, for $a > 1$*

$$\mathrm{RS} = \lim_{N\to\infty} \frac{1}{aN} \log EZ_N^a,$$

where $Z_N = \sum_{\boldsymbol{\sigma}} \exp\big(H_N(\boldsymbol{\sigma}) + h\sum_{i\leq N}\sigma_i\big)$.

It follows from (6.5) (and the results of Sec. 4) that $\liminf_{N\to\infty}(aN)^{-1} \log EZ_N^a \geq \mathrm{RS}$. (This is where the convexity of ξ is required.) The problem is the reverse inequality. In view of (4.17), to prove this reverse inequality, we can assume that H_N is the Hamiltonian (4.2). The arguments do not use that the function ξ is convex.

Proposition 9.5. *There exists a sequence $d_N \to 0$ and a constant K depending on a and ξ only with the following property. If*

$$Z_{N,u} = \sum_{\boldsymbol{\sigma}} \exp\left(H_N(\boldsymbol{\sigma}) + \frac{u}{N^{1/4}}\sum_{i\leq N} h_i\sigma_i + h\sum_{i\leq N}\sigma_i\right), \tag{9.16}$$

then

$$\frac{1}{a}\log EZ_{N,u}^a - \frac{1}{a}\log EZ_{N-1,u}^a \leq \mathrm{RS} + K\big(E'\langle R_{1,2}^2\rangle - (E'\langle R_{1,2}\rangle)^2\big)^{1/2} + d_N, \tag{9.17}$$

where the meaning of E' and $\langle\cdot\rangle$ in (9.17) is as in (9.8).

Proof of Theorem 9.4: Since $\limsup a_N/N \le \limsup(a_N - a_{N-1})$, using the inequalities (9.17) and (9.8), we get, after multiplication by $\sqrt{u}$ and integration that

$$\limsup_{N\to\infty} \int_0^1 \frac{\sqrt{u}}{aN} \log EZ^a_{N,u} du \le \frac{2}{3}\text{RS}. \tag{9.18}$$

Also, by (9.6) and (9.9) we have $\varphi'(u) \ge 0$ so $EZ^a_{N,u} \ge EZ^a_{N,0}$, and (9.18) implies that

$$\limsup_{N\to\infty} \frac{1}{aN} \log EZ^a_N \le \text{RS},$$

completing the proof. □

In summary, we add the perturbation term $uN^{-1/4}\sum_{i\le N} h_i g_i$ that does not affect "the limiting free energy." Yet this term makes the relation $R_{1,2} \simeq$ constant appear out of thin air. Certainly this is less than satisfactory!

Writing $\overline{q} = \overline{q}_{N,u} = E'\langle R_{1,2}\rangle$, we have

$$E'\left\langle R^2_{1,2}\right\rangle - \overline{q}^2 = E'(\langle (R_{1,2} - \overline{q})^2\rangle) \tag{9.19}$$

and Proposition 9.5 really amounts to a kind of cavity computation, that was already done in Sec. 3, but now under a control of the fluctuations of the overlaps. It falls well within the standard techniques. Let us write $\boldsymbol{\varrho} = (\sigma_1, \ldots, \sigma_{N-1})$,

$$\begin{aligned} H(\boldsymbol{\varrho}) = \sum_{p\ge 1} \beta_p \left(\frac{p!}{N^{p-1}}\right)^{1/2} \sum_{i_1<\cdots<i_p\le N-1} g_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p} \\ + \frac{u}{N^{1/4}} \sum_{i\le N-1} h_i\sigma_i + h \sum_{i\le N-1} \sigma_i \end{aligned} \tag{9.20}$$

$$A(\boldsymbol{\varrho}) = \sum_{p\ge 1} \beta_p \left(\frac{p!}{N^{p-1}}\right)^{1/2} \sum_{1\le i_1<\cdots<i_{p-1}\le N-1} g_{i_1\ldots i_{p-1}N}\sigma_{i_1}\cdots\sigma_{i_{p-1}} + \frac{u}{N^{1/4}} h_N, \tag{9.21}$$

so that the Hamiltonian of (9.16) is $H(\boldsymbol{\varrho}) + \sigma_N(A(\boldsymbol{\varrho}) + h)$. Let us write

$$Z = \sum_{\boldsymbol{\varrho}} \exp H(\boldsymbol{\varrho})$$

so that

$$Z_{N,u} = Z\langle 2\text{ch}(A(\boldsymbol{\varrho}) + h)\rangle_0, \tag{9.22}$$

where $\langle\cdot\rangle_0$ denotes an average for the Gibbs measure with Hamiltonian (9.20). Let

$$R^-_{1,2} = R^-(\boldsymbol{\varrho}^1, \boldsymbol{\varrho}^2) = \frac{1}{N} \sum_{i \le N-1} \sigma_i^1 \sigma_i^2$$

(when $\boldsymbol{\varrho}^j = (\sigma_i^j)_{i \le N-1}$).

Lemma 9.6. *We have*

$$E\left(\frac{Z^a}{EZ^a}\langle (R^-_{1,2} - \overline{q})^2\rangle_0\right) \le K\left(E'(\langle (R_{1,2} - \overline{q})^2\rangle) + \frac{1}{N}\right). \tag{9.23}$$

Here and below, K denotes a constant depending on a and ξ only, that need not be the same at each occurrence.

Proof: This is an essentially trivial statement, due to the fact that $EA(\boldsymbol{\varrho})^2 \le K$ for each $\boldsymbol{\varrho}$, and (most importantly) that the randomness of A is independent of the randomness in Z. Using Hölder's inequality, we see first that

$$EZ^a_{N,u} \le 2^a EZ^a \langle \mathrm{ch}^a(A(\boldsymbol{\varrho}) + h)\rangle_0 = 2^a EZ^a \langle E\mathrm{ch}^a(A(\boldsymbol{\varrho}) + h)\rangle_0 \le KEZ^a.$$

Also, assuming for definiteness $a \ge 2$, and using that $Z_{N,u} \ge Z$ by (9.22),

$$E\big(Z^a_{N,u}\langle (R^-_{1,2} - \overline{q})^2\rangle\big) \ge E(Z^{a-2} \sum (R^-_{1,2} - \overline{q})^2 \exp(H(\boldsymbol{\varrho}^1) + H(\boldsymbol{\varrho}^2)$$
$$+ \varepsilon_1(A(\boldsymbol{\varrho}^1) + h) + \varepsilon_2(A(\boldsymbol{\varrho}^2) + h)),$$

where the summation is over all values of $\boldsymbol{\varrho}^1, \boldsymbol{\varrho}^2$ and $\varepsilon_1, \varepsilon_2 = \pm 1$. Integrating in the randomness of $A(\boldsymbol{\varrho})$ and using Jensen's inequality yields

$$E\big(Z^a_{N,u}\langle (R^-_{1,2} - \overline{q})^2\rangle\big) \ge E(Z^a \langle (R^-_{1,2} - \overline{q})^2\rangle_0)$$

and since $(R^-_{1,2} - \overline{q})^2 \le 2(R_{1,2} - \overline{q})^2 + 2/N$, the proof is finished. □

Lemma 9.7. *We have*

$$\frac{1}{a}\log\frac{EZ^a_{N,u}}{EZ^a} \le \log 2 + \frac{1}{2}(\xi'(1) - \xi'(\overline{q})) + \frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{\xi'(\overline{q})} + h)$$
$$+ d_N + KE_0\langle |R^-_{1,2} - \overline{q}|\rangle_0. \tag{9.24}$$

Here and below, (d_N) denotes a sequence with $d_N \to 0$. It need not be the same at each occurrence.

Proof: Using (9.22), we have

$$\frac{1}{a}\log\frac{EZ_{N,u}^a}{EZ^a}=\frac{1}{a}\log E_0 2^a\langle \mathrm{ch}(A(\boldsymbol{\varrho})+h)\rangle_0^a, \tag{9.25}$$

where E_0 denotes integration after change of density Z^a/EZ^a. The important fact is that

$$\forall \boldsymbol{\varrho}^1,\boldsymbol{\varrho}^2,\quad |EA(\boldsymbol{\varrho}^1)A(\boldsymbol{\varrho}^2)-\xi'(R_{1,2}^-)|\le d_N, \tag{9.26}$$

as is seen from (9.21), and thus

$$E_0\langle |EA(\boldsymbol{\varrho}^1)A(\boldsymbol{\varrho}^2)-\xi'(\overline{q})|\rangle \le d_N+KE_0\langle |R_{1,2}^- -\overline{q}|\rangle. \tag{9.27}$$

Let us consider a new independent standard Gaussian r.v. z and

$$\psi(t)=\frac{1}{a}\log E_0\langle \mathrm{ch}(\sqrt{t}A(\boldsymbol{\varrho})+\sqrt{1-t}z\sqrt{\xi'(\overline{q})}+h)\rangle_0^a,$$

so that from (9.25) we have

$$\frac{1}{a}\log\frac{EZ_{N,u}^a}{EZ^a}=\log 2+\psi(1) \tag{9.28}$$

and

$$\psi(0)=\frac{1}{a}\log E\mathrm{ch}^a(z\sqrt{\xi'(\overline{q})}+h). \tag{9.29}$$

It is tedious but straightforward, using integration by parts, to show that, using (9.27),

$$\left|\psi'(t)-\frac{1}{2}(\xi'(1)-\xi'(\overline{q}))\right|\le KE_0\langle |R_{1,2}^- -\overline{q}|\rangle_0+d_N, \tag{9.30}$$

so that, since $\psi(1)-\psi(0)=\int_0^1\psi'(t)\,dt$, (9.24) follows from (9.28) to (9.30). □

Lemma 9.8. *We have*

$$\frac{1}{a}\log\frac{EZ_{N-1,u}^a}{EZ^a}\ge\frac{1}{2}(\theta(1)-\theta(\overline{q}))+\frac{a}{2}\theta(\overline{q})-d_N-KE_0\langle |R_{1,2}^- -\overline{q}|\rangle. \tag{9.31}$$

Proof of Proposition 9.5. Combining (9.24) and (9.31) we see that

$$\begin{aligned}\frac{1}{a}\log EZ_{N,u}^a-\frac{1}{a}\log EZ_{N-1,u}^a &\le \log 2+\frac{1}{2}(\xi(1)-\xi'(\overline{q}))\\ &\quad+\frac{1}{a}\log E\mathrm{ch}^a(h+z\sqrt{\xi'(\overline{q})})\\ &\quad-\frac{1}{2}(a-1)\theta(\overline{q})+d_N+KE_0\langle |R_{1,2}^- -\overline{q}|\rangle_0\\ &\le \mathrm{RS}+d_N+KE_0\langle |R_{1,2}^- -\overline{q}|\rangle_0,\end{aligned}$$

and we conclude using Lemma 9.6. □

Proof of Lemma 9.8. Consider new standard Gaussian r.v. $g'_{i_1\dots i_p}$, h'_i and

$$B(\varrho) = \sum_{p\geq 1} \beta_p (p!)^{1/2} \left(\frac{1}{(N-1)^{(p-1)/2}} - \frac{1}{N^{(p-1)/2}} \right) \sum_{i_1<\cdots<i_p\leq N-1} g'_{i_1\dots i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$
$$+ u \left(\frac{1}{(N-1)^{1/4}} - \frac{1}{N^{1/4}} \right) \sum_{i\leq N-1} h'_i \sigma_i. \tag{9.32}$$

It should be clear (see (4.8)) that

$$\frac{1}{a} \log \frac{E Z^a_{N-1,u}}{E Z^a} = \frac{1}{a} \log E_0 \langle \exp B(\varrho) \rangle_0^a.$$

We deduce from (9.32) that

$$|E B(\varrho^1) B(\varrho^2) - \theta(R^-_{1,2})| \leq d_N,$$

and we proceed as in Lemma 9.7, using the function

$$\psi(t) = \frac{1}{a} \log E_0 \langle \exp(\sqrt{t} B(\varrho) + \sqrt{1-t} z \sqrt{\theta(\overline{q})}) \rangle^a. \qquad \square$$

10. ON THE DOTSENKO-FRANZ-MÉZARD CONJECTURE

In the very interesting paper,[3] the authors argue that for the SK model without external field one should have

$$\forall a < 0, \quad \lim_{N\to\infty} \frac{1}{Na} \log E Z_N^a = \lim_{N\to\infty} \frac{1}{N} E \log Z_N. \tag{10.1}$$

One can hope that eventually one will be able to prove the general Parisi conjecture and to deduce (10.1) from it. In the meantime we are going to prove that (10.1) is a consequence of some likely (but unproven ...) facts about the SK model.

Definition 10.1. *We say that a spin glass system (depending on a parameter N) contains an orthogonal structure if the following occurs. There exists a sequence* $(a_k)_{k\geq 1}$, $a_k > 0$, *such that given* $k_0 \in \mathbb{N}$ *and* $\varepsilon > 0$, *for N large enough (depending on* k_0 *and* ε*) the following event occurs with probability at least* 3/4*:*

$$\forall k \leq k_0 \;\; \exists A_k \subset \{-1, 1\}^N; \;\; G(A_k) \geq a_k \tag{10.2}$$

$$\forall k, l \leq k_0, k \neq l \;\; \langle |R_{1,2}| 1_{\{\sigma^1\in A_k\}} 1_{\{\sigma^2\in A_l\}} \rangle \leq \varepsilon. \tag{10.3}$$

Of course here $G(A_k)$ is the Gibbs measure of A_k, and $R_{1,2} = N^{-1} \sum_{i\leq N} \sigma_i^1 \sigma_i^2$. The sequence (a_k) is permitted to decrease as fast as one wishes. In words, the statement means that one can find (infinitely many) sets of configurations with Gibbs weight bounded below independently of N, that are asymptotically orthogonal, in the sense that the overlap of two configurations in different sets is nearly zero.

Conjecture 10.2. *If $h = 0$, the system governed by the Hamiltonian (3.2) contains an orthogonal structure.*

We do not know how to prove this, even by assuming that the Parisi measure charges zero (or, equivalently, that $E\langle 1_{\{R_{1,2}\simeq 0\}}\rangle$ is bounded below independently of N) and even using the fact that (modulo a small perturbation) the overlaps take essentially (if the Parisi measure charges zero) only finitely many values (see Ref. 9). However (if we understand correctly) Conjecture 10.3 is a consequence of the picture, predicted by the physicists, that the system decomposes in a series of pure states with ultrametric organization as described in Ref. 8, Chapter 4.

Theorem 10.3. *If the system governed by the Hamiltonian (3.2) has an orthogonal structure, then, for any $a < 0$ we have*

$$\lim_{N\to\infty} \left| \frac{1}{N} E \log Z_N - \frac{1}{aN} \log E Z_N^a \right| = 0.$$

To start the proof let $a = -b$, $b > 0$. We first recall that by Hölder's inequality we have

$$\exp E \log \frac{1}{Z} \leq \left(E \frac{1}{Z^b} \right)^{1/b},$$

so that $a^{-1} \log E Z_N^a \leq E \log Z_N$. To get information in the other direction, we have to bound $P(Z_N \leq v)$ from above, in particular when $v \leq e^{-cN}$, where $c = c_N = N^{-1} E \log Z_N$. Given k_0 and $\varepsilon > 0$ let us consider the event Ω' given by (10.2) and (10.3), so that by hypothesis we have $P(\Omega') \geq 3/4$. By concentration of measure (see Ref. 12, Theorem 2.2.4) for some number C depending only of the sequence (β_p) we have $P(\Omega) \geq 1/2$, where

$$\Omega = \Omega' \cap \{Z_N \geq \exp(Nc - C\sqrt{N})\}. \tag{10.4}$$

The randomness of H_N is created by the Gaussian r.v. $g_{i_1 \ldots i_p}$. We denote by $\boldsymbol{g}$ the family of these r.v., so Ω can be thought of as a condition on $\boldsymbol{g}$. We try to relate what happens for two different choices $\boldsymbol{g}$ and $\boldsymbol{g}'$ of these r.v. when $\boldsymbol{g}' \in \Omega$ and

$$d^2(\boldsymbol{g}, \boldsymbol{g}') = \sum (g_{i_1 \ldots i_p} - g'_{i_1 \ldots i_p})^2 \leq u^2 \tag{10.5}$$

(It might help the reader first to understand the proof of Ref. 12, Theorem 2.2.7 as the present proof elaborates the same idea.) We denote by a $'$ the quantities relative to $\boldsymbol{g}'$, so that we have the identity

$$Z_N = Z'_N \langle \mathcal{E} \rangle', \tag{10.6}$$

where

$$\mathcal{E} = \exp\left(\sum_p \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p} \delta_{i_1\ldots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \right),$$

for $\delta_{i_1\ldots i_p} = g_{i_1\ldots i_p} - g'_{i_1\ldots i_p}$. Given a set $A \subset \{-1,1\}^N$ of configurations, by Jensen's inequality we have

$$\langle \mathcal{E} \rangle' \geq \langle 1_A \mathcal{E} \rangle' \geq \langle 1_A \rangle' \exp\left(\sum_p \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p} \delta_{i_1\ldots i_p} \langle \sigma_{i_1} \cdots \sigma_{i_p} \rangle_A \right) \tag{10.7}$$

where

$$\langle f \rangle_A = \frac{\langle f 1_A \rangle'}{\langle 1_A \rangle'}.$$

For $l \leq k_0$, we consider the vector $\boldsymbol{w}_l$ in ℓ^2 of components

$$\frac{\beta_p}{N^{(p-1)/2}} \langle \sigma_{i_1} \cdots \sigma_{i_p} \rangle_{A_l},$$

(where p and $i_1, \ldots i_p$ take all possible values) so that trivially we have

$$||\boldsymbol{w}_l|| \leq \left(N \sum_p \beta_p^2 \right)^{1/2}. \tag{10.8}$$

Also, if $k \neq l$, $k, l \leq k_0$, we have, since $|R_{1,2}| \leq 1$, $\langle 1_{A_k} \rangle' \geq a_k$

$$\begin{aligned} \boldsymbol{w}_k \cdot \boldsymbol{w}_l &= \sum_p \frac{\beta_p^2}{N^{p-1}} \sum_{i_1,\ldots,i_p} \langle \sigma_{i_1} \cdots \sigma_{i_p} \rangle_{A_k} \langle \sigma_{i_1} \cdots \sigma_{i_p} \rangle_{A_l} \\ &= N \sum_p \beta_p^2 \big\langle R_{1,2}^p 1_{A_k}(\boldsymbol{\sigma}^1) 1_{A_l}(\boldsymbol{\sigma}^2) \big\rangle' \frac{1}{\langle 1_{A_k} \rangle' \langle 1_{A_l} \rangle'} \\ &\leq N\varepsilon' \sum_p \beta_p^2 \end{aligned} \tag{10.9}$$

where

$$\varepsilon' = \varepsilon \sup_{k,l \leq k_0} \frac{1}{a_k a_l}.$$

In words, the vectors $(w_l)_{l \leq k_0}$ are nearly orthogonal.

Lemma 10.4. *For each integer k_0, there is a number $\varepsilon_0 > 0$, such that given any vectors $\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{k_0}$ in ℓ^2, if $\|\boldsymbol{y}_k\| \le B$ for $k \le k_0$ and $|\boldsymbol{y}_k \cdot \boldsymbol{y}_l| \le \varepsilon_0 B^2$ for $k \ne l$, $k, l \le k_0$, then for some $l \le k_0$, we have*

$$|\boldsymbol{x} \cdot \boldsymbol{y}_l| \le \frac{2\|\boldsymbol{x}\|B}{\sqrt{k_0}}.$$

Proof: If $\varepsilon_0 = 0$, one can even find l with $|\boldsymbol{x} \cdot \boldsymbol{y}_l| \le \|\boldsymbol{x}\|B/\sqrt{k}$ and the result is obvious by a compactness argument. For those who find that such an argument is out of place here, we give a direct argument. For numbers c_k we have

$$\begin{aligned}
\left\| \sum_{k \le k_0} c_k \boldsymbol{y}_k \right\|^2 &= \sum_k c_k^2 \|\boldsymbol{y}_k\|^2 + \sum_{k \ne l} c_k c_l \boldsymbol{y}_k \cdot \boldsymbol{y}_l \\
&\le B^2 \sum_k c_k^2 + \sum_{k \ne l} |c_k||c_l| \varepsilon_0 B^2 \\
&\le B^2 \left(\sum_k c_k^2 + \varepsilon_0 \left(\sum_k |c_k| \right)^2 \right) \\
&\le B^2 (1 + \varepsilon_0 k_0) \left(\sum_k c_k^2 \right) \\
&\le 2B^2 \sum_k c_k^2, \qquad (10.10)
\end{aligned}$$

provided that $\varepsilon_0 k_0 \le 1$ and using the Cauchy-Schwartz inequality. Now

$$\begin{aligned}
\boldsymbol{x} \cdot \left(\sum_k c_k \boldsymbol{y}_k \right) &= \sum_k c_k \boldsymbol{x} \cdot \boldsymbol{y}_k \le \|\boldsymbol{x}\| \left\| \sum_k c_k \boldsymbol{y}_k \right\| \\
&\le \|\boldsymbol{x}\| \sqrt{2} B \sqrt{\sum_k c_k^2},
\end{aligned}$$

by (10.10) and taking $c_k = \boldsymbol{x} \cdot \boldsymbol{y}_k \left(\sum_l (\boldsymbol{x} \cdot \boldsymbol{y}_l)^2 \right)^{-1/2}$ shows that $\sum_{k \le k_0} (\boldsymbol{x} \cdot \boldsymbol{y}_k)^2 \le 2B^2 ||\boldsymbol{x}||^2$. □

It follows from this lemma and (10.8) that we can find $l \le k_0$ with

$$|\boldsymbol{\delta} \cdot \boldsymbol{w}_l| \le 2C_0 \frac{\|\boldsymbol{\delta}\| \sqrt{N}}{\sqrt{k_0}} \le 2C_0 \frac{\sqrt{N}}{\sqrt{k_0}} u, \qquad (10.11)$$

where $C_0 = \sqrt{\sum \beta_p^2}$, and using (10.5) in the second inequality. When applied to $A = A_l$ (10.7) yields, using (10.11),

$$\langle \mathcal{E} \rangle' \geq \langle 1_{A_l} \rangle' \exp(\boldsymbol{w}_l \cdot \boldsymbol{\delta}) \geq a_l \exp\left(-2C_0 \frac{\sqrt{N}}{\sqrt{k_0}} u\right).$$

Since $Z'_N \geq \exp(Nc - C\sqrt{N})$ by (10.4), setting $\gamma_{k_0} = \log \inf_{k \leq k_0} a_k$, we see from (10.6) that

$$Z_N \geq \exp\left(Nc - 2u\sqrt{N}\frac{C_0}{\sqrt{k_0}} - C\sqrt{N} + \gamma_{k_0}\right). \tag{10.12}$$

This lower bound is valid for $u = d(\boldsymbol{g}, \Omega)$, where d is the distance in l^2. Now by concentration of measure (as in Ref. 12, Lemma 2.2.11) we get

$$P(d(\boldsymbol{g}, \Omega) \geq u + L) \leq 2\exp\left(-\frac{u^2}{4}\right).$$

Thus, if $C_1 = 2C_0$, $C_2 = C + 2L + 1$, for large N we see from (10.12) that

$$P\left(Z_N < \exp\left(Nc - uC_1\sqrt{\frac{N}{k_0}} - C_2\sqrt{N}\right)\right) \leq 2\exp\left(-\frac{u^2}{4}\right),$$

or, equivalently, for $v > 0$, and a new constant C_3

$$P\left(Z_N < \exp\left(N\left(c - v - \frac{C_2}{\sqrt{N}}\right)\right)\right) \leq 2\exp\left(-\frac{k_0 v^2 N}{C_3}\right). \tag{10.13}$$

We use the formula

$$E\frac{1}{Z_N^b} = \int_0^\infty P\left(\frac{1}{Z_N^b} \geq t\right) dt = bN \int_{\mathbb{R}} P(Z_N \leq \exp xN)\exp(-bxN)\, dx, \tag{10.14}$$

by setting $t = \exp(-bxN)$. We have

$$\begin{aligned} &bN \int_{x \geq c - C_2/\sqrt{N}} P(Z_N \leq \exp xN)\exp(-bxN)\, dx \\ &\quad \leq bN \int_{x \geq c - C_2/\sqrt{N}} \exp(-bxN) \leq \exp Nb\left(-c + \frac{C_2}{\sqrt{N}}\right) \end{aligned} \tag{10.15}$$

Also, setting $x = c - v - C_2/\sqrt{N}$,

$$\begin{aligned} &bN \int_{x \leq c - C_2\sqrt{N}} P(Z_N \leq \exp xN)\exp(-bxN)\, dx \\ &\quad \leq \exp bN\left(-c + \frac{C_2}{\sqrt{N}}\right) \int_{v \geq 0} P\left(Z_N \leq \exp N\left(c - v - \frac{C_2}{\sqrt{N}}\right)\right) \exp(bvN)\, dv \\ &\quad \leq 2\exp bN\left(-c + \frac{C_2}{\sqrt{N}}\right) \int_{v \geq 0} \exp\left(-\frac{k_0 N v^2}{C_3}\right) \exp(bvN)\, dv. \end{aligned} \tag{10.16}$$

We conclude from (10.14) to (10.16) that for N large we have

$$\frac{1}{bN}\log E\frac{1}{Z_N^b}\leq -c+\frac{C_4 b}{\sqrt{k_0}}$$

and since k_0 is arbitrary, this finishes the proof of Theorem 10.3.

11. THE SPHERICAL MODEL

In this section we study the spherical model with Hamiltonian (3.2) (and external field h). The space of configurations is now the sphere

$$S_N=\left\{\boldsymbol{\sigma}\in\mathbb{R}^N;\ \sum_{i\leq N}\sigma_i^2=N\right\}.$$

We denote by λ_N the uniform measure on S_N.

Consider $a<1$ and

$$\begin{aligned} a=m_0<m_1<\cdots<m_k\leq 1\\ q_0=0<q_1<\cdots<q_k\leq q_{k+1}=1. \end{aligned}\tag{11.1}$$

There is nothing to change in the proof of Guerra's bound for the case of Ising spins to get

$$\frac{1}{Na}\log EZ_N^a\leq\frac{X_0}{N}-\frac{1}{2}\sum_{0\leq l\leq k}m_l(\theta(q_{l+1})-\theta(q_l))\tag{11.2}$$

where

$$Z_N=\int_{S_N}\exp\left(H_N(\boldsymbol{\sigma})+h\sum_{i\leq N}\sigma_i\right)d\lambda_N(\boldsymbol{\sigma})$$

and X_l is defined recursively by

$$X_{k+1}=\log\left(\int\exp\left(\sum_{i\leq N}\sigma_i\left(h+\sum_{0\leq p\leq k}z_p^i\right)\right)d\lambda_N(\boldsymbol{\sigma})\right),$$

$$X_l=\frac{1}{m_l}\log E_l\exp m_l X_{l+1},$$

where z_p^i are independent Gaussian with $E(z_p^i)^2=\xi'(q_{p+1})-\xi'(q_p)$.

When $a>1$, if one reverses the inequalities in (11.1), one obtains the reverse inequality in (10.3). There is nothing to change to the work of Ref. 14 to show that

$$\lim_{N\to\infty}\frac{X_0}{N}=\inf_b W(x,b),\tag{11.3}$$

where

$$W(x,b) = \frac{1}{2}\left(\frac{h^2}{b-d(0)} + \int_0^1 \frac{\xi''(s)}{b-d(s)}ds + b - 1 - \log b\right)$$

for

$$d(q) = \int_q^1 \xi''(s)x(s)\,ds,$$

where $x(s) = m_l$ for $q_l < s \le q_{l+1}$. The infimum in (11.3) is over all values of b for which $b > \sup\{d(s); 0 \le s \le 1\}$. The bound (11.2) can then be rewritten as

$$\inf_b \left(-\frac{1}{2}\int_0^1 \theta'(q)x(q)\,dq + W(x,b)\right), \tag{11.4}$$

so that

$$\frac{1}{Na}\log EZ_N^a \le \inf_{b,x}\left(-\frac{1}{2}\int_0^1 \theta'(q)x(q)\,dq + W(x,b)\right). \tag{11.5}$$

Crisanti and Sommers[(2)] have found a remarkable way to rewrite this formula. They found that the right hand side of (11.5) is

$$\inf_x \mathcal{P}(x) \tag{11.6}$$

where

$$\mathcal{P} = \frac{1}{2}\left(\int_0^1 x(q)\xi'(q)\,dq + h^2\widehat{x}(0) + \int_0^{\widehat{q}} \frac{dq}{\widehat{x}(q)} + \log(1-\widehat{q})\right), \tag{11.7}$$

where x is a non decreasing function $[0,1] \to [a,1]$ such that $x(\widehat{q}) = 1$ for some $\widehat{q} < 1$, and $\widehat{x}(q) = \int_q^1 x(s)\,ds > 0$ for each $q > 0$. The quantity (11.6) is quite easier technically to study that the right-hand side of (11.5).

Moreover not only do we have equality of the right-hand side of (11.5) and of (11.6), but when we constraint the function x to take only k given values, the infimum is obtained for the same function x. This is a consequence of the proofs in Sec. 4 of Ref. 14. (With the notations there, when $\mathbf{q}'$ and b' are such that $A(\mathbf{q}', b') = \inf A(\mathbf{q}, b)$, we have $A(\mathbf{q}', b') = B(\mathbf{q}') = \inf_{\mathbf{q}} B(\mathbf{q})$).

Theorem 11.1.
a) If x minimizes (11.6) over all non decreasing functions $[0,1] \to [a,1]$ *(such that* $\widehat{x}(q) > 0$ *for each q or, equivalently,* $\widehat{x}(0) > 0$*) then x is constant equal to a in an interval starting at* 0 *and is* ≥ 0 *after that.*
b) Same as a), when one moreover requires that x takes at most k different values.

Proof: If x is such that $\mathcal{P}(x)$ is minimum, for any other non decreasing function $y : [0,1] \to [a,1]$ with $\widehat{y} > 0$, we have

$$\mathcal{P}((1-\varepsilon)x + \varepsilon y) = \mathcal{P}(x + \varepsilon(y - x)) \geq \mathcal{P}(x).$$

Let $z = y - x$. Writing that the derivative at $\varepsilon \geq 0$ of $\mathcal{P}(x + \varepsilon z)$ is ≥ 0, we have

$$\int_0^1 z(q)\xi'(q)\,dq + h^2 \int_0^1 z(q)\,dq - \int_0^{\widehat{q}} \frac{dq}{\widehat{x}(q)^2} \int_q^1 z(s)\,ds \geq 0. \tag{11.8}$$

Now

$$\int_0^{\widehat{q}} \frac{dq}{\widehat{x}(q)^2} \int_q^1 z(s)\,ds = \int_0^1 z(q)\,dq \int_0^{\min(\widehat{q},q)} \frac{1}{\widehat{x}(s)^2} ds$$

so that (11.8) implies that

$$\int_0^1 z(q)F(q)\,dq \geq 0 \tag{11.9}$$

for

$$F(q) = \xi'(q) + h^2 - \int_0^{\min(\widehat{q},q)} \frac{1}{\widehat{x}(s)^2} ds.$$

Consider the positive measure ν' such that

$$\nu'([0,q]) = \frac{y(q) - a}{1 - a}, \tag{11.10}$$

so that

$$\int_0^1 y(q)F(q)\,dq = \int_0^1 (a + (1-a)\nu'([0,q]))F(q)\,dq$$

and

$$\begin{aligned}\int_0^1 \nu'([0,q])F(q)\,dq &= \int_0^1 F(q)\,dq \int_0^q d\nu'(s)\\ &= \int_0^1 d\nu'(s) \int_s^1 F(q)\,dq.\end{aligned}$$

Consider the measure ν defined as in (11.10) but with $x(q)$ instead of $y(q)$. It follows from (11.9) that

$$\int_0^1 d\nu'(s) \int_0^s F(q)\,dq \leq \int_0^1 d\nu(s) \int_0^s F(q)\,dq,$$

whatever the choice of y. This implies that ν is supported by the set of points s where $\int_0^s F(q)\,dq$ is maximum. Now, for $q < \widehat{q}$,

$$F'(q) = \xi''(q) - \frac{1}{\widehat{x}(q)^2}$$

and $\widehat{x}(q)' = -x(q)$ is > 0 when $x(q) < 0$, so that $\widehat{x}(q)$ increases and $-\widehat{x}(q)^{-2}$ also increases. Thus $F(q)$ is convex on the largest interval where $x(q) \leq 0$. In particular it can have at most 2 zeros in this interval. Consider

$$q_1 = \sup\{q : \ x(q) < 0\},$$

so that q_1 is in the support of ν. Suppose, if possible, that there exists $0 \leq q_0 < q_1$ in the support of ν. Since $\int_0^s F(q)\,dq$ is maximum at each point of the support of ν, we have

$$F(q_0) = F(q_1) = \int_{q_0}^{q_1} F(q)\,dq = 0. \tag{11.11}$$

(This is true even if $q_0 = 0$, because $F(0) \geq 0$, and if $\int_0^s F(q)\,dq$ has a maximum at $s = 0$, we must have $F(0) = 0$.) But (11.11) implies that F has a zero between q_0 and q_1, so that F has at least 3 zeros in the interval $[0, q_1]$, which is impossible. Thus the support of ν does not meet the interval $[0, q_1[$, i.e. x is constant equal to a in this interval.

This proves a). The proof of b) is very similar. One observe simply that if x takes the value m_l on the interval $[q_l, q_{l+1}[$, the fact that one cannot decrease $\mathcal{P}(x)$ by a small variation of q_l means that $F(q_l) = 0$, and the fact that one cannot decrease $\mathcal{P}(x)$ by a small variation of m_l means that $\int_{q_l}^{q_{l+1}} F(q)\,dq = 0$. □

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